Micromorphic media

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Plan

Notations

Cartesian bases: reference basis $(\underline{\mathbf{E}}_K)_{K=1,2,3}$, current basis $(\underline{\mathbf{e}}_i)_{i=1,2,3}$

$$\underline{\mathbf{A}} = A_i \, \underline{\mathbf{e}}_i, \quad \underset{\sim}{\mathbf{A}} = A_{ij} \, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad \underset{\simeq}{\mathbf{A}} = \underset{\eqsim}{\mathbf{A}} = \underline{\underline{\underline{\mathbf{A}}}} = A_{ijk} \, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k, \quad \underset{\approx}{\mathbf{A}}$$

symmetric and skew–symmetric parts $\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{A}}^s + \underset{\sim}{\mathbf{A}}^a$

tensor products

$$\underline{\mathbf{a}} \otimes \underline{\mathbf{b}} = a_i b_j \, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad \underset{\sim}{\mathbf{A}} \otimes \underset{\sim}{\mathbf{B}} = A_{ij} B_{kl} \, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k \otimes \underline{\mathbf{e}}_l$$

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nabla operators

$$\nabla_x = \;_{,i} \, \underline{\mathbf{e}}_i, \quad \nabla_X = \;_{,K} \, \underline{\mathbf{E}}_K$$

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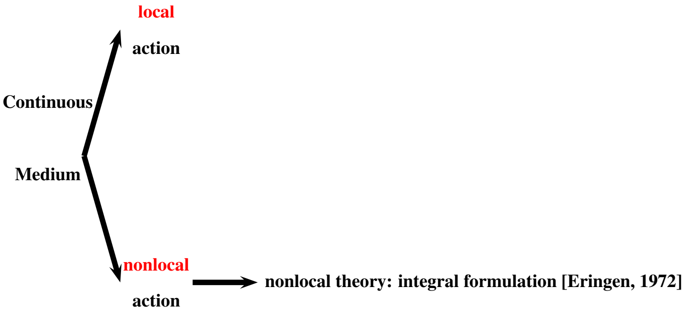
Mechanics of generalized continua

Principle of local action: *the stress state at a point* $\underline{\mathbf{X}}$ *depends on variables defined at this point only*

[Truesdell, Toupin, 1960] [Truesdell, Noll, 1965]



Mechanics of generalized continua

Simple material: *A material is simple at the particle* $\underline{\mathbf{X}}$ *if and only if its response to deformations homogeneous in a neighborhood of* $\underline{\mathbf{X}}$ *determines uniquely its response to every deformation at* $\underline{\mathbf{X}}$ *.*

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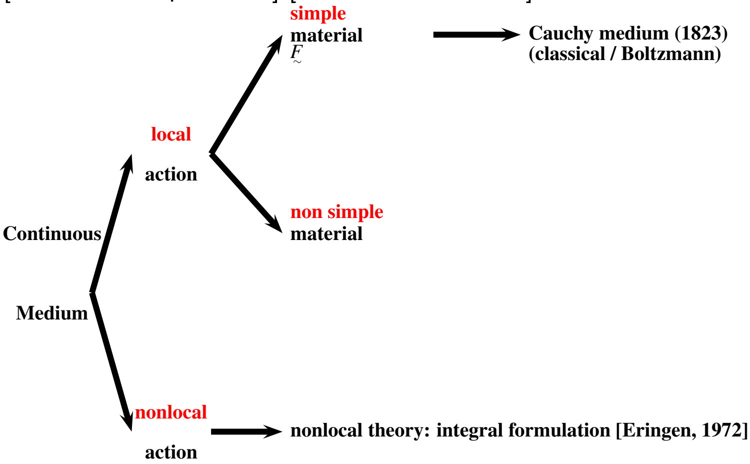

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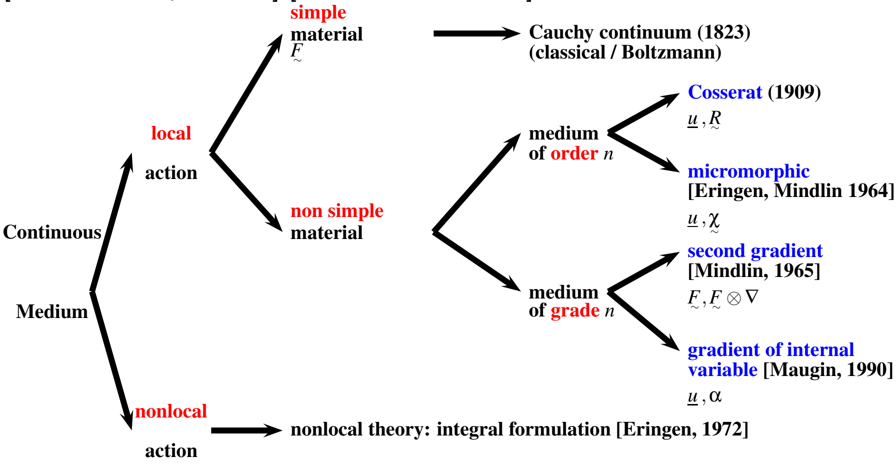

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Kinematics of micromorphic media

- Degrees of freedom of the theory

$$DOF := \{ \quad \underline{\mathbf{u}}, \quad \underset{\sim}{\chi} \quad \}$$

  - ⋆ displacement $\underline{\mathbf{u}}(\underline{\mathbf{X}}, t)$ and microdeformation $\underset{\sim}{\chi}(\underline{\mathbf{X}}, t)$ of the material point $\underline{\mathbf{X}}$
  - ⋆ current position of the material point $\underline{\mathbf{x}} = \Phi(\underline{\mathbf{X}}, t) = \underline{\mathbf{X}} + \underline{\mathbf{u}}(\underline{\mathbf{X}}, t)$
  - ⋆ deformation of a triad of directors attached to the material point $\underline{\xi}^{\,i}(\underline{\mathbf{X}}) = \underset{\sim}{\chi}(\underline{\mathbf{X}}) \cdot \underline{\mathbf{\Xi}}^{\,i}$

- Polar decomposition of the generally incompatible microdeformation field $\underset{\sim}{\chi}(\underline{\mathbf{X}}, t)$

$$\underset{\sim}{\chi} = \underset{\sim}{\mathbf{R}}^{\sharp} \cdot \underset{\sim}{\mathbf{U}}^{\sharp}$$

internal constraints

  - ⋆ Cosserat medium $\quad \underset{\sim}{\chi} \equiv \underset{\sim}{\mathbf{R}}^{\sharp}$
  - ⋆ Microstrain medium $\quad \underset{\sim}{\chi} \equiv \underset{\sim}{\mathbf{U}}^{\sharp}$
  - ⋆ Second gradient medium $\quad \underset{\sim}{\chi} \equiv \underset{\sim}{\mathbf{F}}$

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Directors in materials

trièdre directeur in a single crystal: 3 lattice vectors

"Les directeurs ne subissent pas la même transformation que les lignes matérielles. C'est en cela que le milieu plastique diffère du milieu continu classique. On doit le concevoir un peu comme un milieu de Cosserat." [Mandel, 1973]

Kinematics of micromorphic media

- velocity field $\underline{\mathbf{v}}(\underline{\mathbf{x}}, t) := \dot{\underline{\mathbf{u}}}(\Phi^{-1}(\underline{\mathbf{x}}, t), t)$
- deformation gradient $\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}} + \underline{\mathbf{u}} \otimes \boldsymbol{\nabla}_X$
- velocity gradient $\underline{\mathbf{v}} \otimes \boldsymbol{\nabla}_x = \dot{\underset{\sim}{\mathbf{F}}} \cdot \underset{\sim}{\mathbf{F}}^{-1}$
- microdeformation rate $\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}$
- Lagrangian microdeformation gradient $\underset{\approx}{\mathbf{K}} := \underset{\sim}{\chi}^{-1} \cdot \underset{\sim}{\chi} \otimes \boldsymbol{\nabla}_X$
- gradient of the microdeformation rate tensor

$$(\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}) \otimes \boldsymbol{\nabla}_x = \underset{\sim}{\chi} \cdot \dot{\underset{\approx}{\mathbf{K}}} : (\underset{\sim}{\chi}^{-1} \boxtimes \underset{\sim}{\mathbf{F}}^{-1})$$

$$(\dot{\chi}_{iL}\chi^{-1}_{Lj})_{,k} = \chi_{iP}\dot{K}_{PQR}\chi^{-1}_{Qj}F^{-1}_{Rk}$$

[Eringen, 1999]

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Power of internal forces

- Model variables according to a first gradient theory
  $MODEL = \{ \quad \underline{\mathbf{v}}, \quad \underline{\mathbf{v}} \otimes \boldsymbol{\nabla}_x, \quad \dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}, \quad (\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}) \otimes \boldsymbol{\nabla}_x \quad \}$
- Virtual power of internal forces of a subdomain $\mathcal{D} \subset \mathcal{B}$

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) = \int_{\mathcal{D}} p^{(i)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) \, dv$$

- The virtual power density of internal forces is a linear form on the fields of virtual modeling variables

$$\begin{aligned} p^{(i)} &= \underset{\sim}{\sigma} : (\dot{\underset{\sim}{F}} \cdot \underset{\sim}{F}^{-1}) + \underset{\sim}{s} : (\dot{\underset{\sim}{F}} \cdot \underset{\sim}{F}^{-1} - \dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}) + \underset{\approx}{M} \,\vdots\, ((\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}) \otimes \boldsymbol{\nabla}_x) \\ &= \underset{\sim}{\sigma} : (\dot{\underset{\sim}{F}} \cdot \underset{\sim}{F}^{-1}) + \underset{\sim}{s} : (\underset{\sim}{\chi} \cdot (\underset{\sim}{\chi}^{-1} \cdot \underset{\sim}{F})^{\cdot} \cdot \underset{\sim}{F}^{-1}) + \underset{\approx}{M} \,\vdots\, \left(\underset{\sim}{\chi} \cdot \dot{\underset{\tilde{}}{\underline{K}}} : (\underset{\sim}{\chi}^{-1} \boxtimes \underset{\sim}{F}^{-1}) \right) \end{aligned}$$

relative deformation rate $\dot{\underset{\sim}{F}} \cdot \underset{\sim}{F}^{-1} - \dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}$
relative deformation $\underset{\sim}{\Upsilon} := \underset{\sim}{\chi}^{-1} \cdot \underset{\sim}{F}$

- The virtual power density of internal forces is invariant with respect to a Euclidean change of observer $\Rightarrow \underset{\sim}{\sigma}$ is symmetric [Germain, 1973]

Power of internal forces

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Power of contact forces

- Application of Gauss theorem to the power of internal forces

$$\begin{aligned}\int_{\mathcal{D}} p^{(i)}\, dV &= \int_{\partial\mathcal{D}} \underline{\mathbf{v}}^* \cdot (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}) \cdot \underline{\mathbf{n}}\; dS + \int_{\partial\mathcal{D}} (\dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) : \underset{\approx}{\mathbf{M}} \cdot \underline{\mathbf{n}}\; ds \\ &- \int_{\mathcal{D}} \underline{\mathbf{v}}^* \cdot (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}) \cdot \boldsymbol{\nabla}_x\, dV - \int_{\mathcal{D}} (\dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) : \underset{\approx}{\mathbf{M}} \cdot \boldsymbol{\nabla}_x\, dv\end{aligned}$$

  The form of the previous boundary integral dictates the form of the

- power of contact forces acting on the boundary $\partial\mathcal{D}$ of the subdomain $\mathcal{D} \subset \mathcal{B}$

$$\mathcal{P}^{(c)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) = \int_{\partial\mathcal{D}} p^{(c)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1})\, ds$$

$$p^{(c)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) = \underline{\mathbf{t}} \cdot \underline{\mathbf{v}}^* + \underset{\sim}{\mathbf{m}} : (\dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1})$$

  simple traction $\underline{\mathbf{t}}$, double traction $\underset{\sim}{\mathbf{m}}$

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- power of contact forces acting on the boundary $\partial\mathcal{D}$ of the subdomain $\mathcal{D} \subset \mathcal{B}$

$$\mathcal{P}^{(c)}(\underline{\mathbf{v}}^{\,*}, \dot{\underset{\sim}{\chi}}^{*} \cdot \underset{\sim}{\chi}^{*-1}) = \int_{\partial\mathcal{D}} p^{(c)}(\underline{\mathbf{v}}^{\,*}, \dot{\underset{\sim}{\chi}}^{*} \cdot \underset{\sim}{\chi}^{*-1})\, ds$$

$$p^{(c)}(\underline{\mathbf{v}}^{\,*}, \dot{\underset{\sim}{\chi}}^{*} \cdot \underset{\sim}{\chi}^{*-1}) = \underline{\mathbf{t}} \cdot \underline{\mathbf{v}}^{\,*} + \underset{\sim}{\mathbf{m}} : (\dot{\underset{\sim}{\chi}}^{*} \cdot \underset{\sim}{\chi}^{*-1})$$

simple traction $\underline{\mathbf{t}}$, double traction $\underset{\sim}{\mathbf{m}}$

Power of forces acting at a distance

$$\mathcal{P}^{(e)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) = \int_{\mathcal{D}} p^{(e)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1})\, dv$$

$$p^{(e)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) = \underline{\mathbf{f}} \cdot \underline{\mathbf{v}}^* + \underset{\sim}{\mathbf{p}} : (\dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1})$$

simple body forces $\underline{\mathbf{f}}$, double body forces $\underset{\sim}{\mathbf{p}}$

more general triple volume forces could be introduced according to [Germain, 1973]

Principle of virtual power

In the static case, $\forall \underline{\mathbf{v}}^*, \forall \underset{\sim}{\chi}^*, \forall \mathcal{D} \subset \mathcal{B}$,

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) = \mathcal{P}^{(c)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) + \mathcal{P}^{(e)}(\underline{\mathbf{v}}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1})$$

[Germain, 1973]

Principle of virtual power

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which leads to

$$\int_{\partial\mathcal{D}} \underline{\mathbf{v}}^* \cdot (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}) \cdot \underline{\mathbf{n}} \; ds + \int_{\partial\mathcal{D}} (\dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{-1}) : \underset{\approx}{\mathbf{M}} \cdot \underline{\mathbf{n}} \; ds$$

$$- \int_{\mathcal{D}} \underline{\mathbf{v}}^* \cdot ((\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}) \cdot \boldsymbol{\nabla}_x + \underline{\mathbf{f}}) \, dv - \int_{\mathcal{D}} (\dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1}) : (\underset{\approx}{\mathbf{M}} \cdot \boldsymbol{\nabla}_x + \underset{\sim}{\mathbf{s}} + \underset{\sim}{\mathbf{p}}) \, dv = 0$$

Balance and boundary conditions

The application of the principle of virtual power leads to the

- balance of momentum equation (static case)

$$(\underset{\sim}{\sigma} + \underset{\sim}{\mathbf{s}}) \cdot \boldsymbol{\nabla}_x + \underline{\mathbf{f}} = 0, \quad \forall \underline{\mathbf{x}} \in \mathcal{B}$$

- balance of generalized moment of momentum equation (static case)

$$\underset{\approx}{\mathbf{M}} \cdot \boldsymbol{\nabla}_x + \underset{\sim}{\mathbf{s}} + \underset{\sim}{\mathbf{p}} = 0, \quad \forall \underline{\mathbf{x}} \in \mathcal{B}$$

- boundary conditions

$$(\underset{\sim}{\sigma} + \underset{\sim}{\mathbf{s}}) \cdot \underline{\mathbf{n}} = \underline{\mathbf{t}}, \quad \forall \underline{\mathbf{x}} \in \partial\mathcal{B}$$

$$\underset{\approx}{\mathbf{M}} \cdot \underline{\mathbf{n}} = \underset{\sim}{\mathbf{m}}, \quad \forall \underline{\mathbf{x}} \in \partial\mathcal{B}$$

Plan

A hierarchy of higher order continua

name	number of DOF	DOF (finite case)	DOF (infinitesimal case)	references
Cauchy	3	$\underline{\mathbf{u}}$	$\underline{\mathbf{u}}$	[Cauchy, 1823]
micromorphic	12	$\underline{\mathbf{u}}, \quad \underset{\sim}{\chi}$	$\underline{\mathbf{u}}, \quad \underset{\sim}{\chi}^s + \underset{\sim}{\chi}^a$	[Eringen, 1964] [Mindlin, 1964]

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Cosserat	6	$\underline{\mathbf{u}}\,,\quad \underset{\sim}{\mathbf{R}}$	$\underline{\mathbf{u}}\,,\quad \underline{\boldsymbol{\Phi}}$	[Kafadar, Eringen, 1976]
micromorphic	12	$\underline{\mathbf{u}}\,,\quad \underset{\sim}{\boldsymbol{\chi}}$	$\underline{\mathbf{u}}\,,\quad \underset{\sim}{\boldsymbol{\chi}}^s + \underset{\sim}{\boldsymbol{\chi}}^a$	[Eringen, 1964] [Mindlin, 1964]

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microstretch	7	$\underline{\mathbf{u}}, \quad \chi, \quad \underset{\sim}{\mathbf{R}}$	$\underline{\mathbf{u}}, \quad \chi, \quad \underline{\boldsymbol{\Phi}}$	[Eringen, 1990]
micromorphic	12	$\underline{\mathbf{u}}, \quad \underset{\sim}{\boldsymbol{\chi}}$	$\underline{\mathbf{u}}, \quad \underset{\sim}{\boldsymbol{\chi}}^s + \underset{\sim}{\boldsymbol{\chi}}^a$	[Eringen, 1964] [Mindlin, 1964]

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Cosserat	6	$\underline{\mathbf{u}}\,,\quad \underset{\sim}{\mathbf{R}}$	$\underline{\mathbf{u}}\,,\quad \underline{\boldsymbol{\Phi}}$	[Kafadar, Eringen, 1976]
microstretch	7	$\underline{\mathbf{u}}\,,\quad \chi,\quad \underset{\sim}{\mathbf{R}}$	$\underline{\mathbf{u}}\,,\quad \chi,\quad \underline{\boldsymbol{\Phi}}$	[Eringen, 1990]
microstrain	9	$\underline{\mathbf{u}}\,,\quad \underset{\sim}{\mathbf{C}}^\sharp$	$\underline{\mathbf{u}}\,,\quad {}^{\chi}\underset{\sim}{\varepsilon}$	[Forest, Sievert, 2006]
micromorphic	12	$\underline{\mathbf{u}}\,,\quad \underset{\sim}{\chi}$	$\underline{\mathbf{u}}\,,\quad \underset{\sim}{\chi}^s + \underset{\sim}{\chi}^a$	[Eringen, 1964] [Mindlin, 1964]

Some well–knwon generalized continua

	1D	2D	3D
higher order	Timoshenko/Cosserat beam	Mindlin plate/shell	micromorphic continuum
higher grade	Euler–Bernoulli beam	Love–Kirchhoff plate	second gradient medium

Some words on rotations

- Rotation

$$\underset{\sim}{\mathbf{R}} \cdot \underset{\sim}{\mathbf{R}}^T = \underset{\sim}{\mathbf{R}}^T \cdot \underset{\sim}{\mathbf{R}} = \underset{\sim}{\mathbf{1}}, \quad \det \underset{\sim}{\mathbf{R}} = 1$$

- Representation of finite rotations

$$\underset{\sim}{\mathbf{R}} = \exp(-\underset{\approx}{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\Phi}})$$

rotation vector $\underline{\boldsymbol{\Phi}} = \theta \underline{\mathbf{n}}$

$$\underset{\sim}{\mathbf{R}} = \cos\theta \underset{\sim}{\mathbf{1}} + \frac{1-\cos\theta}{\theta^2} \underline{\boldsymbol{\Phi}} \otimes \underline{\boldsymbol{\Phi}} - \frac{\sin\theta}{\theta} \underset{\approx}{\boldsymbol{\epsilon}} . \underline{\boldsymbol{\Phi}}$$

The skew symmetric part of $\underset{\sim}{\mathbf{R}}$ gives the rotation axis

$$\overset{\times}{\underline{\mathbf{R}}} = -\frac{1}{2} \underset{\approx}{\boldsymbol{\epsilon}} : \underset{\sim}{\mathbf{R}} = -\frac{1}{2} \epsilon_{klm} R_{lm} \underline{\mathbf{e}}_k = \sin\theta \underline{\mathbf{n}}$$

- Small rotations

$$\| \underset{\sim}{\mathbf{R}} - \underset{\sim}{\mathbf{1}} \| \ll 1$$

$$\underset{\sim}{\mathbf{R}} \simeq \underset{\sim}{\mathbf{1}} - \underset{\approx}{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\Phi}}, \quad \underset{\sim}{\mathbf{R}}^a \simeq -\underset{\approx}{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\Phi}}$$

Strain measures for the nonlinear Cosserat continuum

$\underset{\sim}{\chi} \equiv \underset{\sim}{\mathbf{R}}^{\sharp}$

- Strain and relative rotation in a single Lagrangian tensor

$$\underset{\sim}{\boldsymbol{\Upsilon}} = \underset{\sim}{\mathbf{R}}^{\sharp T} \cdot \underset{\sim}{\mathbf{F}} = \underbrace{\underset{\sim}{\mathbf{R}}^{\sharp T} \cdot \underset{\sim}{\mathbf{R}}}_{\text{relative rotation}} \cdot \underset{\sim}{\mathbf{U}}$$

- Cosserat rotation vector $\quad \underline{\boldsymbol{\Phi}} = \sin\theta\, \underline{\mathbf{n}}, \quad \underset{\sim}{\mathbf{R}}^{\sharp} = \exp(-\underset{\approx}{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\Phi}})$
- The third rank rotation gradient can be reduced to the second rank curvature tensor:

$$d\underline{\boldsymbol{\xi}}^{\,i} = d\underset{\sim}{\mathbf{R}}^{\sharp} \cdot \underline{\boldsymbol{\Xi}}^{\,i} = \underbrace{d\underset{\sim}{\mathbf{R}}^{\sharp} \cdot \underset{\sim}{\mathbf{R}}^{\sharp T}}_{\text{skew}-\text{symmetric}} \cdot \underline{\boldsymbol{\xi}}^{\,i}$$

$$d\underset{\sim}{\mathbf{R}}^{\sharp} \cdot \underset{\sim}{\mathbf{R}}^{\sharp T} = -\underset{\approx}{\boldsymbol{\epsilon}} \cdot d\underline{\boldsymbol{\Phi}}, \quad d\underline{\boldsymbol{\Phi}} = -\frac{1}{2}\underset{\approx}{\boldsymbol{\epsilon}} : (d\underset{\sim}{\mathbf{R}}^{\sharp} \cdot \underset{\sim}{\mathbf{R}}^{\sharp T})$$

$$d\underline{\boldsymbol{\Phi}} = \underset{\sim}{\mathbf{K}} \cdot d\underline{\mathbf{X}}, \quad \underset{\sim}{\mathbf{K}} = \frac{1}{2}\underset{\approx}{\boldsymbol{\epsilon}} : (\underset{\sim}{\mathbf{R}}^{\sharp} \cdot (\underset{\sim}{\mathbf{R}}^{\sharp T} \otimes \boldsymbol{\nabla}_X))$$

Strain measures for the nonlinear Cosserat continuum

Details of the calculation

$$
\begin{aligned}
d\Phi_i &= -\frac{1}{2}\epsilon_{ijk} dR^\sharp_{jM} R^\sharp_{kM} \\
&= -\frac{1}{2}\epsilon_{ijk} R^\sharp_{jM,N} R^\sharp_{kM} dX_N \\
&= \frac{1}{2}\epsilon_{ikj} R^\sharp_{kM} R^{\sharp T}_{Mj,N} dX_N \\
d\underline{\mathbf{\Phi}} &= \frac{1}{2}\underline{\underline{\epsilon}} : (\underset{\sim}{\mathbf{R}}^\sharp \cdot (\underset{\sim}{\mathbf{R}}^{\sharp T} \otimes \boldsymbol{\nabla}_X))
\end{aligned}
$$

The third rank rotation gradient can be reduced to a second rank **curvature tensor** Lagrangean curvature tensor

$$
\underset{\sim}{\mathbf{K}}^\sharp = \underset{\sim}{\mathbf{R}}^{\sharp T} \cdot \underset{\sim}{\mathbf{K}} = \frac{1}{2}\underset{\sim}{\mathbf{R}}^{\sharp T} \cdot \underline{\underline{\epsilon}} : (\underset{\sim}{\mathbf{R}}^\sharp \cdot (\underset{\sim}{\mathbf{R}}^{\sharp T} \otimes \boldsymbol{\nabla}_X))
$$

Plan

Energy balance

- **kinetic energy**

$$\mathcal{K} := \frac{1}{2}\int_{\mathcal{D}} \rho \underline{\mathbf{v}}\,.\,\underline{\mathbf{v}} + (\dot{\underset{\sim}{\boldsymbol{\chi}}}\cdot\underset{\sim}{\boldsymbol{\chi}}^{-1}) : (\dot{\underset{\sim}{\boldsymbol{\chi}}}\cdot\underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\underset{\sim}{\mathbf{i}})\, dv$$

  Eringen's tensor of microinertia $\underset{\sim}{\mathbf{i}}$ (symmetric)

- **power of external forces**

$$\mathcal{P} := \mathcal{P}^c + \mathcal{P}^e = \int_{\partial\mathcal{D}} \underline{\mathbf{t}}\,.\,\underline{\mathbf{v}} + \underset{\sim}{\mathbf{m}} : (\dot{\underset{\sim}{\boldsymbol{\chi}}}\cdot\underset{\sim}{\boldsymbol{\chi}}^{-1})\, ds + \int_{\mathcal{D}} \underline{\mathbf{f}}\,.\,\underline{\mathbf{v}} + \underset{\sim}{\mathbf{p}} : (\dot{\underset{\sim}{\boldsymbol{\chi}}}\cdot\underset{\sim}{\boldsymbol{\chi}}^{-1})\, dv$$

- **internal energy** $\mathcal{E}$, mass density e of internal energy

$$\mathcal{E} := \int_{\mathcal{D}} \rho e(\underline{\mathbf{x}}, t)\, dv$$

- **heat** supply $\mathcal{Q}$ to the system in the form of contact heat supply $h(\underline{\mathbf{x}}, t, \partial\mathcal{D})$ and volume heat supply $\rho r(\underline{\mathbf{x}}, t)$

$$\mathcal{Q} := \int_{\partial\mathcal{D}} h\, ds + \int_{\mathcal{D}} \rho r\, dv$$

  heat flux vector q $\quad h(\underline{\mathbf{x}}, \underline{\mathbf{n}}, t) = -\mathbf{q}(\underline{\mathbf{x}}, t).\underline{\mathbf{n}}$

Energy balance

- **kinetic energy**

$$\mathcal{K} := \frac{1}{2}\int_{\mathcal{D}} \rho \underline{\mathbf{v}} \,.\, \underline{\mathbf{v}} + (\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}) : (\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1} \cdot \underset{\sim}{\mathbf{i}})\, dv$$

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Energy balance

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Energy balance

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$$\mathcal{Q} := \int_{\partial\mathcal{D}} h\, ds + \int_{\mathcal{D}} \rho r\, dv$$

  heat flux vector $\underline{\mathbf{q}}$ $\qquad h(\underline{\mathbf{x}}, \underline{\mathbf{n}}, t) = -\underline{\mathbf{q}}(\underline{\mathbf{x}}, t).\underline{\mathbf{n}}$

Energy principle

$$\dot{\mathcal{E}} + \dot{\mathcal{K}} = \mathcal{P} + \mathcal{Q}$$

Taking the **theorem of kinetic energy** into account,

$$\dot{\mathcal{K}} = \mathcal{P}^i + \mathcal{P}^e + \mathcal{P}^c$$

where, in the absence of discontinuities,

$$\mathcal{P}^i = -\int_{\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}} + \underset{\sim}{\mathbf{s}} : (\dot{\underset{\sim}{\mathbf{F}}} \cdot \underset{\sim}{\mathbf{F}}^{-1} - \dot{\underset{\sim}{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}) + \underset{\simeq}{\mathbf{M}} \vdots \left((\dot{\underset{\sim}{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}) \otimes \boldsymbol{\nabla}_x \right) \, dv$$

is the **power of internal forces**, the first principle can be rewritten as

$$\dot{\mathcal{E}} = -\mathcal{P}^i + \mathcal{Q}$$

$$\begin{aligned} \int_{\mathcal{D}} \rho \dot{e} \, dv \quad &= \quad \int_{\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}} + \underset{\sim}{\mathbf{s}} : (\dot{\underset{\sim}{\mathbf{F}}} \cdot \underset{\sim}{\mathbf{F}}^{-1} - \dot{\underset{\sim}{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}) + \underset{\simeq}{\mathbf{M}} \vdots (\dot{\underset{\sim}{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}) \otimes \boldsymbol{\nabla}_x dv \\ &- \quad \int_{\partial \mathcal{D}} \underline{\mathbf{q}} \, . \underline{\mathbf{n}} \, ds + \int_{\mathcal{D}} \rho r \, dv \end{aligned}$$

Local formulation of the energy principle

From the global formulation for any sub–domain $\mathcal{D} \subset \mathcal{B}_t$...

$$\int_{\mathcal{D}} \rho \dot{e} \, dv = \int_{\mathcal{D}} p^{(i)} \, dv - \int_{\partial \mathcal{D}} \underline{\mathbf{q}} \, . \underline{\mathbf{n}} \; ds + \int_{\mathcal{D}} \rho r \, dv$$

... to the local formulation at a regular point of $\mathcal{B}_t$

$$\rho \dot{e} = p^{(i)} - \operatorname{div} \underline{\mathbf{q}} + \rho r$$

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}} + \underset{\sim}{\mathbf{s}} : (\underset{\sim}{\dot{\mathbf{F}}} \cdot \underset{\sim}{\mathbf{F}}^{-1} - \underset{\sim}{\dot{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}) + \underset{\simeq}{\mathbf{M}} \,\vdots\, \left((\underset{\sim}{\dot{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}) \otimes \boldsymbol{\nabla}_x \right)$$

Lagrangian formulation of the energy principle

Lagrangian representation in continuum thermodynamics

$$e(\underline{\mathbf{x}}\,, t) = e_0(\underline{\mathbf{X}}\,, t), \quad \underline{\mathbf{Q}}\,(\underline{\mathbf{X}}\,, t) = J\underset{\sim}{\mathbf{F}}^{-1}.\underline{\mathbf{q}}$$

From the global formulation for any sub–domain $\mathcal{D}_0 \subset \mathcal{B}_0$...

$$\int_{\mathcal{D}_0} \rho_0 \dot{e}_0 \, dV = \int_{\mathcal{D}_0} \underset{\sim}{\mathbf{\Pi}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\sim}{\mathbf{S}} : (\underset{\sim}{\chi}^{-1} \cdot \underset{\sim}{\mathbf{F}})^{\cdot} + \underset{\simeq}{\mathbf{M}}_0 \vdots \dot{\underset{\simeq}{\mathbf{K}}} \, dV - \int_{\partial\mathcal{D}_0} \underline{\mathbf{Q}} \,.\underline{\mathbf{N}} \, dS + \int_{\mathcal{D}_0} \rho_0 r_0 \, dV$$

... to the local formulation at a regular point of $\mathcal{B}_0$

$$\rho_0 \dot{e}_0 = \underset{\sim}{\mathbf{\Pi}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\sim}{\mathbf{S}} : (\underset{\sim}{\chi}^{-1} \cdot \underset{\sim}{\mathbf{F}})^{\cdot} + \underset{\simeq}{\mathbf{M}}_0 \vdots \dot{\underset{\simeq}{\mathbf{K}}} - \mathrm{Div}\, \underline{\mathbf{Q}} + \rho_0 r_0$$

Piola–Kirchhoff tensors

$$\underset{\sim}{\mathbf{\Pi}} = J\underset{\sim}{\mathbf{F}}^{-1} \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underset{\sim}{\mathbf{F}}^{-T}, \quad \underset{\sim}{\mathbf{S}} = J\,\underset{\sim}{\chi}^{T} \cdot \underset{\sim}{\mathbf{s}} \cdot \underset{\sim}{\mathbf{F}}^{-T}, \quad \underset{\simeq}{\mathbf{M}}_0 = J\,\underset{\sim}{\chi}^{T} \cdot \underset{\simeq}{\mathbf{M}} : (\underset{\sim}{\chi}^{-T} \boxtimes \underset{\sim}{\mathbf{F}}^{-T})$$

Entropy principle

- entropy of the system / mass entropy density

$$\mathcal{S}(\mathcal{D}) = \int_{\mathcal{D}} \rho s \, dv$$

- entropy supply

$$\varphi(\mathcal{D}) = -\int_{\partial\mathcal{D}} \frac{\underline{\mathbf{q}}}{T} . \underline{\mathbf{n}} \, ds + \int_{\mathcal{D}} \frac{\rho r}{T} \, dv$$

- global formulation of the entropy principle for any sub–domain $\mathcal{D} \subset \mathcal{B}_t$

$$\dot{\mathcal{S}}(\mathcal{D}) - \varphi(\mathcal{D}) \geq 0$$

$$\frac{d}{dt} \int_{\mathcal{D}} \rho s \, dv + \int_{\partial\mathcal{D}} \frac{\underline{\mathbf{q}}}{T} . \underline{\mathbf{n}} \, ds - \int_{\mathcal{D}} \rho \frac{r}{T} \, dv \geq 0$$

Lagrangian formulation of the entropy principle

Lagrangian description in continuum thermodynamics

$$s(\underline{\mathbf{x}}\,,t) = s_0(\underline{\mathbf{X}}\,,t), \quad \underline{\mathbf{Q}}\,(\underline{\mathbf{X}}\,,t) = J\underset{\sim}{\mathbf{F}}^{-1}.\underline{\mathbf{q}}$$

From the global formulation valid for any sub–domain $\mathcal{D}_0 \subset \mathcal{B}_0$...

$$\frac{d}{dt}\int_{\mathcal{D}_0} \rho_0 s_0(\underline{\mathbf{X}}\,,t)\,dV + \int_{\partial\mathcal{D}_0} \frac{\underline{\mathbf{Q}}}{T}.\underline{\mathbf{N}}\,dS + \int_{\mathcal{D}_0} \rho_0 \frac{r_0}{T}\,dV \geq 0$$

... to the local formulation at a regular point $\mathcal{B}_0$

$$\rho_0 \dot{s}_0 + \mathrm{Div}\,\frac{\underline{\mathbf{Q}}}{T} - \rho_0 \frac{r_0}{T} \geq 0$$

Dissipation

- State variables for elastic materials

$$STATE = \{\underset{\sim}{\mathbf{E}} := (\underset{\sim}{\mathbf{F}}^T \cdot \underset{\sim}{\mathbf{F}} - \underset{\sim}{\mathbf{1}})/2, \quad \underset{\sim}{\Upsilon} := \underset{\sim}{\chi}^{-1} . \underset{\sim}{\mathbf{F}}, \quad \underset{\approx}{\mathbf{K}} := \underset{\sim}{\chi}^{-1} . (\underset{\sim}{\chi} \otimes \nabla_X), \ T\}$$

- functions of state: internal energy $e_0(\underset{\sim}{\mathbf{E}}, \underset{\sim}{\Upsilon}, \underset{\approx}{\mathbf{K}}, s_0)$
  Helmholtz free energy $\psi_0(\underset{\sim}{\mathbf{E}}, \underset{\sim}{\Upsilon}, \underset{\approx}{\mathbf{K}}, T) = e_0 - T s_0$
- Clausius–Duhem inequality (volume dissipation rate D)

$$D = \underset{\sim}{\Pi} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\sim}{\mathbf{S}} : \dot{\underset{\sim}{\Upsilon}} + \underset{\approx}{\mathbf{M}}_0 \vdots \dot{\underset{\approx}{\mathbf{K}}} - \rho_0(\dot{\psi}_0 + \dot{T} s_0) - \underline{\mathbf{Q}} . \frac{\mathrm{Grad}\, T}{T} \geq 0$$

Dissipation

- functions of state: internal energy $e_0(\underset{\sim}{\mathbf{E}}, \underset{\sim}{\boldsymbol{\Upsilon}}, \underset{\simeq}{\mathbf{K}}, s_0)$
  Helmholtz free energy $\psi_0(\underset{\sim}{\mathbf{E}}, \underset{\sim}{\boldsymbol{\Upsilon}}, \underset{\simeq}{\mathbf{K}}, T) = e_0 - Ts_0$
- Clausius–Duhem inequality (volume dissipation rate D)

$$D = \underset{\sim}{\boldsymbol{\Pi}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\sim}{\mathbf{S}} : \dot{\underset{\sim}{\boldsymbol{\Upsilon}}} + \underset{\sim}{\mathbf{M}}_0 \vdots \dot{\underset{\simeq}{\mathbf{K}}} - \rho_0(\dot{\psi}_0 + \dot{T} s_0) - \underline{\mathbf{Q}} \, . \frac{\mathrm{Grad}\; T}{T} \geq 0$$

- Elastic materials

$$\dot{\psi}_0 = \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{E}}} : \dot{\underset{\sim}{\mathbf{E}}} + \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}} : \dot{\underset{\sim}{\boldsymbol{\Upsilon}}} + \frac{\partial \psi_0}{\partial \underset{\simeq}{\mathbf{K}}} \vdots \dot{\underset{\simeq}{\mathbf{K}}} + \frac{\partial \psi_0}{\partial T} \dot{T}$$

$$\begin{aligned} D &= (\underset{\sim}{\boldsymbol{\Pi}} - \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{E}}}) : \dot{\underset{\sim}{\mathbf{E}}} + (\underset{\sim}{\mathbf{S}} - \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}}) : \dot{\underset{\sim}{\boldsymbol{\Upsilon}}} + (\underset{\simeq}{\mathbf{M}}_0 - \rho_0 \frac{\partial \psi_0}{\partial \underset{\simeq}{\mathbf{K}}}) : \dot{\underset{\simeq}{\mathbf{K}}} \\ &- \rho_0 (\frac{\partial \psi_0}{\partial T} + s_0) \dot{T} - \underline{\mathbf{Q}} \, . \frac{\mathrm{Grad}\; T}{T} \geq 0 \end{aligned}$$

State laws for hyperelastic materials

- hyperelastic relations

$$\underset{\sim}{\mathbf{\Pi}} = \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{E}}}, \quad s_0 = -\frac{\partial \psi_0}{\partial T}$$

$$\underset{\sim}{\mathbf{S}} = \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{\Upsilon}}}, \quad \underset{\simeq}{\mathbf{M}}_0 = \rho_0 \frac{\partial \psi_0}{\partial \underset{\simeq}{\mathbf{K}}}$$

ψ_0 is also called **elastic potential** (vanishing intrinsic dissipation)

- thermal dissipation

$$D = -\underline{\mathbf{Q}} \, . \frac{\mathrm{Grad}\ T}{T} = -\frac{\rho_0}{\rho} \underline{\mathbf{q}} \, .\mathrm{grad}\ T \geq 0$$

Fourier law *(thermal constitutive equation)*

$$\underline{\mathbf{Q}} = -\underset{\sim}{\mathbf{K}}(\underset{\sim}{\mathbf{E}}, \underset{\sim}{\mathbf{\Upsilon}}, \underset{\simeq}{\mathbf{K}}, T).\mathrm{Grad}\ T$$

there is no thermal potential (total dissipation)

Plan

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Linearized strain measures

- Small strains

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{R}} \cdot \underset{\sim}{\mathbf{U}}, \quad \underset{\sim}{\mathbf{U}} = \underset{\sim}{\mathbf{1}} + \underset{\sim}{\varepsilon}, \quad \|\underset{\sim}{\varepsilon}\| \ll 1$$

$$\underset{\sim}{\chi} = \underset{\sim}{\mathbf{R}}^\sharp \cdot \underset{\sim}{\mathbf{U}}^\sharp, \quad \underset{\sim}{\mathbf{U}}^\sharp = \underset{\sim}{\mathbf{1}} + \underset{\sim}{\chi}^s, \quad \|\underset{\sim}{\chi}^s\| \ll 1$$

- Small rotations

$$\underset{\sim}{\mathbf{R}} \simeq \underset{\sim}{\mathbf{1}} + \underset{\sim}{\omega}, \quad \underset{\sim}{\omega} = \frac{1}{2}(\underline{\mathbf{u}} \otimes \boldsymbol{\nabla}_X - \boldsymbol{\nabla}_X \otimes \underline{\mathbf{u}}) \quad \|\underset{\sim}{\omega}\| \ll 1$$

$$\underset{\sim}{\mathbf{R}}^\sharp \simeq \underset{\sim}{\mathbf{1}} + \underset{\sim}{\chi}^a = \underset{\sim}{\mathbf{1}} - \underset{\approx}{\boldsymbol{\epsilon}} \cdot \underline{\boldsymbol{\Phi}}, \quad \|\underline{\boldsymbol{\Phi}}\| \ll 1$$

- linearized strain measures $\quad \underset{\sim}{\mathbf{E}} \simeq \underset{\sim}{\varepsilon} = \frac{1}{2}(\underline{\mathbf{u}} \otimes \boldsymbol{\nabla}_X + \boldsymbol{\nabla}_X \otimes \underline{\mathbf{u}})$

$$\underset{\sim}{\boldsymbol{\Upsilon}} = (\underset{\sim}{\mathbf{1}} + \underset{\sim}{\chi}^s + \underset{\sim}{\chi}^a)^{-1} \cdot (\underset{\sim}{\mathbf{1}} + \underset{\sim}{\varepsilon} + \underset{\sim}{\omega}) \simeq \underset{\sim}{\mathbf{1}} + \underbrace{\underset{\sim}{\varepsilon} - \underset{\sim}{\chi}^s}_{\text{relative strain}} + \underbrace{\underset{\sim}{\omega} - \underset{\sim}{\chi}^a}_{\text{relative rotation}}$$

$$\underset{\approx}{\mathbf{K}} \simeq \underset{\sim}{\chi} \otimes \boldsymbol{\nabla}$$

Plan

Linear Cosserat theory

- Degrees of freedom

$$DOF = \{\underline{\mathbf{u}}\,, \quad \underline{\mathbf{\Phi}}\,\}$$

- Cosserat strain measures: relative deformation and curvature tensor

$$\underset{\sim}{\mathbf{e}} = \underline{\mathbf{u}} \otimes \boldsymbol{\nabla} + \underset{\approx}{\underline{\boldsymbol{\epsilon}}} \cdot \underline{\mathbf{\Phi}}$$

$$\underset{\sim}{\boldsymbol{\kappa}} = \underline{\mathbf{\Phi}} \otimes \boldsymbol{\nabla}$$

- Generally non–symmetric force stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$ and couple stress tensor $\underset{\sim}{\mathbf{M}}$
  Balance of momentum: $\operatorname{div} \underset{\sim}{\boldsymbol{\sigma}} + \underline{\mathbf{f}} = 0$
  Balance of moment of momentum: $\operatorname{div} \underset{\sim}{\mathbf{M}} + 2\,\underline{\overset{\times}{\boldsymbol{\sigma}}} + \underline{\mathbf{c}} = 0$
  body couples $\underline{\mathbf{c}}$
- Boundary conditions

$$\underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\mathbf{n}} = \underline{\mathbf{t}}\,, \quad \underset{\sim}{\mathbf{M}} \cdot \underline{\mathbf{n}} = \underline{\mathbf{m}}\,, \quad \forall \underline{\mathbf{x}} \in \partial\Omega$$

Linear Cosserat elasticiy

- Elastic potential

$$\rho\psi(\underset{\sim}{\mathbf{e}}, \underset{\sim}{\boldsymbol{\kappa}}) = \frac{1}{2}\underset{\sim}{\mathbf{e}} : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{e}} + \underset{\sim}{\mathbf{e}} : \underset{\approx}{\mathbf{D}} : \underset{\sim}{\boldsymbol{\kappa}} + \frac{1}{2}\underset{\sim}{\boldsymbol{\kappa}} : \underset{\approx}{\mathbf{A}} : \underset{\sim}{\boldsymbol{\kappa}}$$

centro–symmetric materials: $\underset{\approx}{\mathbf{D}} = 0$

Generalized Hooke's laws: $\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{e}}, \quad \underset{\sim}{\mathbf{M}} = \underset{\approx}{\mathbf{A}} : \underset{\sim}{\boldsymbol{\kappa}}$

- Linear isotropic elasticity (6 elastic moduli)

$$\begin{aligned} \underset{\sim}{\boldsymbol{\sigma}} &= \lambda(\mathrm{trace}\,\underset{\sim}{\mathbf{e}})\,\underset{\sim}{\mathbf{1}} + 2\mu\,\underset{\sim}{\mathbf{e}}^s + 2\mu_c\,\underset{\sim}{\mathbf{e}}^a \\ \underset{\sim}{\mathbf{M}} &= \alpha(\mathrm{trace}\,\underset{\sim}{\mathbf{e}})\,\underset{\sim}{\mathbf{1}} + 2\beta\,\underset{\sim}{\boldsymbol{\kappa}}^s + 2\gamma\,\underset{\sim}{\boldsymbol{\kappa}}^a \end{aligned}$$

- Elastic stability

$$3\lambda + 2\mu \geq 0, \quad \mu \geq 0, \quad \mu_c \geq 0$$

$$3\alpha + 2\beta \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0$$

physical units? [Nowacki, 1986, Cao et al., 2013]

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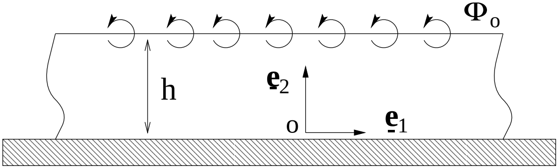
Simple glide problem for the Cosserat continuum



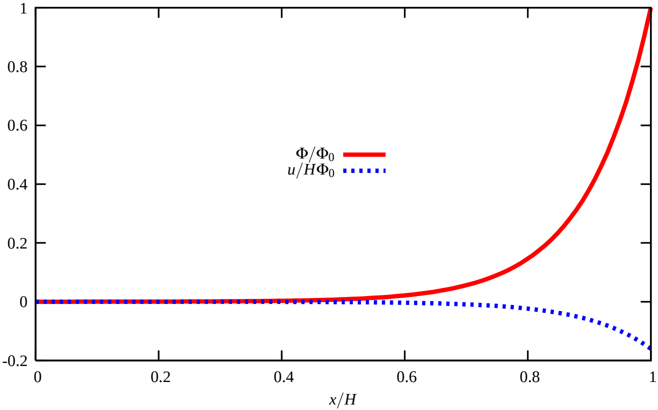
$$\underline{\mathbf{u}} = \begin{pmatrix} u(x_2) \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{\Phi}} = \begin{pmatrix} 0 \\ 0 \\ \Phi(x_2) \end{pmatrix}$$

$$\underset{\sim}{\mathbf{e}} = \underset{\sim}{\nabla}\underline{\mathbf{u}} + \underset{\sim}{\underline{\epsilon}}.\underline{\mathbf{\Phi}} = \begin{pmatrix} 0 & u' + \Phi & 0 \\ -\Phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{\boldsymbol{\kappa}} = \underline{\mathbf{\Phi}} \otimes \underline{\nabla} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Phi' & 0 \end{pmatrix}$$

Simple glide problem for the Cosserat continuum

$\mu = 30000$ MPa, $\beta = 500$ MPa.mm^2, $\mu_c = 100000$ MPa,
$\ell_c/h \simeq 0.1$



Consider limit cases: $\ell_c \to 0, \quad \mu_c \to \infty$

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Finite deformation of elastoviscoplastic micromorphic media

- strain measures

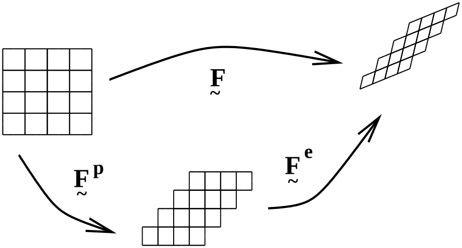
$$STRAIN = \{\underset{\sim}{\mathbf{C}} := \underset{\sim}{\mathbf{F}}^T \cdot \underset{\sim}{\mathbf{F}}, \quad \underset{\sim}{\boldsymbol{\Upsilon}} := \underset{\sim}{\boldsymbol{\chi}}^{-1} . \underset{\sim}{\mathbf{F}}, \quad \underset{\approx}{\mathbf{K}} := \underset{\sim}{\boldsymbol{\chi}}^{-1} . (\underset{\sim}{\boldsymbol{\chi}} \otimes \boldsymbol{\nabla}_X)\}$$

- multiplicative decomposition of the deformation gradient

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{F}}^e \cdot \underset{\sim}{\mathbf{F}}^p = \underset{\sim}{\mathbf{R}}^e \cdot \underset{\sim}{\mathbf{U}}^e \cdot \underset{\sim}{\mathbf{F}}^p$$

according to [Mandel, 1973]. The

uniqueness of the decomposition requires the suitable definition of directors.



Finite deformation of elastoviscoplastic micromorphic media

- strain measures

$$STRAIN = \{ \underset{\sim}{\mathbf{C}} := \underset{\sim}{\mathbf{F}}^T \cdot \underset{\sim}{\mathbf{F}}, \quad \underset{\sim}{\boldsymbol{\Upsilon}} := \underset{\sim}{\chi}^{-1} . \underset{\sim}{\mathbf{F}}, \quad \underset{\approx}{\mathbf{K}} := \underset{\sim}{\chi}^{-1} . (\underset{\sim}{\chi} \otimes \boldsymbol{\nabla}_X) \}$$

- multiplicative decomposition of the deformation gradient

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{F}}^e \cdot \underset{\sim}{\mathbf{F}}^p = \underset{\sim}{\mathbf{R}}^e \cdot \underset{\sim}{\mathbf{U}}^e \cdot \underset{\sim}{\mathbf{F}}^p$$

according to [Mandel, 1973]. The uniqueness of the decomposition requires the suitable definition of directors.

- multiplicative decomposition of the microdeformation

$$\underset{\sim}{\chi} = \underset{\sim}{\chi}^e \cdot \underset{\sim}{\chi}^p = \underset{\sim}{\mathbf{R}}^{e\sharp} \cdot \underset{\sim}{\mathbf{U}}^{e\sharp} \cdot \underset{\sim}{\chi}^p$$

according to [Forest and Sievert, 2003, Forest and Sievert, 2006]. The uniqueness of the decomposition requires the suitable definition of directors.

Finite deformation of elastoviscoplastic micromorphic media

- strain measures

  $STRAIN = \{\underset{\sim}{\mathbf{C}} := \underset{\sim}{\mathbf{F}}^T \cdot \underset{\sim}{\mathbf{F}}, \quad \underset{\sim}{\boldsymbol{\Upsilon}} := \underset{\sim}{\chi}^{-1} . \underset{\sim}{\mathbf{F}}, \quad \underset{\approx}{\mathbf{K}} := \underset{\sim}{\chi}^{-1} . (\underset{\sim}{\chi} \otimes \boldsymbol{\nabla}_X)\}$

- additive decomposition of the micro–deformation gradient

$$\underset{\approx}{\mathbf{K}} = \underset{\approx}{\mathbf{K}}^e + \underset{\approx}{\mathbf{K}}^p$$

  according to [Sansour, 1998].

Plan

Thermodynamics of micromorphic continua

- Local equation of energy

$$\rho\dot{\epsilon} \;=\; p^{(i)} \;-\; \underline{\mathbf{q}}\,.\boldsymbol{\nabla}_x \;+\; r$$

- Second principle

$$\rho\dot{\eta} \;+\; \left(\frac{\underline{\mathbf{q}}}{T}\right).\boldsymbol{\nabla}_x \;-\; \frac{r}{T} \;\geq\; 0$$

$$p^{(i)} - \rho\dot{\Psi} - \eta\dot{T} \;-\; \frac{\underline{\mathbf{q}}}{T}.(\boldsymbol{\nabla}_x T) \;\geq\; 0$$

- State variables and Helmholtz free energy ?

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : (\dot{\underset{\sim}{\mathbf{F}}}\cdot\underset{\sim}{\mathbf{F}}^{-1}) + \underset{\sim}{\mathbf{s}} : (\underset{\sim}{\boldsymbol{\chi}}\cdot(\underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\underset{\sim}{\mathbf{F}})^{\boldsymbol{\cdot}}\cdot\underset{\sim}{\mathbf{F}}^{-1}) + \underset{\approx}{\mathbf{M}} \;\vdots\; \left(\underset{\sim}{\boldsymbol{\chi}}\cdot\dot{\underset{\simeq}{\mathbf{K}}} : (\underset{\sim}{\boldsymbol{\chi}}^{-1}\boxtimes\underset{\sim}{\mathbf{F}}^{-1})\right)$$

Tracking the suitable state variables

$$J\underset{\sim}{\sigma} : \underset{\sim}{\mathbf{L}} = J\underset{\sim}{\sigma} : \dot{\underset{\sim}{\mathbf{F}}} \cdot \underset{\sim}{\mathbf{F}}^{-1} = J\underset{\sim}{\sigma} : (\dot{\underset{\sim}{\mathbf{F}}}^{e} \cdot \underset{\sim}{\mathbf{F}}^{e-1} + \underset{\sim}{\mathbf{F}}^{e}\dot{\underset{\sim}{\mathbf{F}}}^{p} \cdot \underset{\sim}{\mathbf{F}}^{p-1} \cdot \underset{\sim}{\mathbf{F}}^{e-1})$$

$$\begin{aligned}
J\underset{\sim}{\sigma} : \dot{\underset{\sim}{\mathbf{F}}}^{e} \cdot \underset{\sim}{\mathbf{F}}^{e-1} &= \frac{J}{2}\underset{\sim}{\sigma} : \left(\dot{\underset{\sim}{\mathbf{F}}}^{e} \cdot \underset{\sim}{\mathbf{F}}^{e-1} + \underset{\sim}{\mathbf{F}}^{e-T} \cdot \dot{\underset{\sim}{\mathbf{F}}}^{eT}\right) \\
&= \frac{J}{2}\underset{\sim}{\sigma} : \underset{\sim}{\mathbf{F}}^{e-T} \cdot (\underset{\sim}{\mathbf{F}}^{eT} \cdot \dot{\underset{\sim}{\mathbf{F}}}^{e} + \dot{\underset{\sim}{\mathbf{F}}}^{eT} \cdot \underset{\sim}{\mathbf{F}}^{e}) \cdot \underset{\sim}{\mathbf{F}}^{e-1} \\
&= \frac{J}{2}\underset{\sim}{\sigma} : \underset{\sim}{\mathbf{F}}^{e-T} \cdot (\underset{\sim}{\mathbf{F}}^{eT} \cdot \underset{\sim}{\mathbf{F}}^{e})^{\cdot} \cdot \underset{\sim}{\mathbf{F}}^{e-1} \\
&= \underset{\sim}{\mathbf{\Pi}}^{e} : \dot{\underset{\sim}{\mathbf{E}}}^{e}
\end{aligned}$$

$$\underset{\sim}{\mathbf{\Pi}}^{e} = J\underset{\sim}{\mathbf{F}}^{e-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{\mathbf{F}}^{e-T}, \quad \underset{\sim}{\mathbf{E}}^{e} = \frac{1}{2}(\underset{\sim}{\mathbf{F}}^{eT} \cdot \underset{\sim}{\mathbf{F}}^{e} - \underset{\sim}{\mathbf{1}})$$

Tracking the suitable state variables

Relative elastic strain

$$\underset{\sim}{\Upsilon} = \underset{\sim}{\chi}^{-1} \cdot \underset{\sim}{\mathbf{F}} = \underset{\sim}{\chi}^{p} \cdot \underbrace{\underset{\sim}{\chi}^{e-1} \cdot \underset{\sim}{\mathbf{F}}^{e}}_{\underset{\sim}{\Upsilon}^{e}} \cdot \underset{\sim}{\mathbf{F}}^{p}$$

$$\begin{aligned} \underset{\sim}{\chi} \cdot \dot{\underset{\sim}{\Upsilon}} \cdot \underset{\sim}{\mathbf{F}}^{-1} &= \underset{\sim}{\chi} \cdot (\underset{\sim}{\chi}^{e} \cdot \dot{\underset{\sim}{\Upsilon}}^{e} \cdot \underset{\sim}{\mathbf{F}}^{e-1} - \underset{\sim}{\chi}^{p-1} \cdot \dot{\underset{\sim}{\chi}}^{p} \cdot \underset{\sim}{\chi}^{p-1} \cdot \underset{\sim}{\Upsilon}^{e} \cdot \underset{\sim}{\mathbf{F}}^{p} + \underset{\sim}{\chi}^{p-1} \cdot \underset{\sim}{\Upsilon}^{e} \cdot \dot{\underset{\sim}{\mathbf{F}}}^{p}) \cdot \underset{\sim}{\mathbf{F}}^{-1} \\ &= \underset{\sim}{\chi}^{e} \cdot \dot{\underset{\sim}{\Upsilon}}^{e} \cdot \underset{\sim}{\mathbf{F}}^{e-1} - \underset{\sim}{\chi}^{e} \cdot \dot{\underset{\sim}{\chi}}^{p} \cdot \underset{\sim}{\chi}^{p-1} \cdot \underset{\sim}{\Upsilon}^{e} \cdot \underset{\sim}{\mathbf{F}}^{e-1} + \underset{\sim}{\chi}^{e} \cdot \underset{\sim}{\Upsilon}^{e} \cdot \dot{\underset{\sim}{\mathbf{F}}}^{p} \cdot \underset{\sim}{\mathbf{F}}^{p-1} \cdot \underset{\sim}{\mathbf{F}}^{e-1} \end{aligned}$$

$$J \underset{\sim}{\mathbf{s}} : (\underset{\sim}{\chi}^{e} \cdot \dot{\underset{\sim}{\Upsilon}}. \cdot \underset{\sim}{\mathbf{F}}^{e-1}) = \underbrace{J \underset{\sim}{\chi}^{eT} \cdot \underset{\sim}{\mathbf{s}} \cdot \underset{\sim}{\mathbf{F}}^{e-T}}_{\underset{\sim}{\mathbf{S}}^{e}} : \dot{\underset{\sim}{\Upsilon}}^{e}$$

Elastic part of the microdeformation gradient

$$J \underset{\sim}{\mathbf{M}} \vdots (\underset{\sim}{\chi} \cdot \dot{\underset{\simeq}{\mathbf{K}}}^{e} : (\underset{\sim}{\chi}^{-1} \boxtimes \underset{\sim}{\mathbf{F}}^{-1}))$$

Hyperelastic state laws

- State variables

$$STATE := \{\underset{\sim}{\mathbf{E}}^e, \quad \underset{\sim}{\boldsymbol{\Upsilon}}^e, \quad \underset{\simeq}{\mathbf{K}}^e, \quad q, \quad T\}$$

set of internal variables q

- Clausius–Duhem inequality

$$(\underset{\sim}{\boldsymbol{\Pi}}^e - \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{E}}^e}) : \dot{\underset{\sim}{\mathbf{E}}}^e + (\underset{\sim}{\mathbf{S}}^e - \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}^e}) : \dot{\underset{\sim}{\boldsymbol{\Upsilon}}}^e + (\underset{\simeq}{\mathbf{M}}_0 - \rho_0 \frac{\partial \psi_0}{\partial \underset{\simeq}{\mathbf{K}}^e}) \vdots \dot{\underset{\simeq}{\mathbf{K}}}^e$$

$$-\rho_0 \frac{\partial \psi_0}{\partial q} \dot{q} - \rho_0 (\frac{\partial \psi_0}{\partial T} + s_0) \dot{T} + D^{res} \geq 0$$

Hyperelastic state laws

state laws including hyperelastic relationships

$$J\,\underset{\sim}{\boldsymbol{\sigma}} = 2\underset{\sim}{\mathbf{F}}^e \cdot \rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{C}}^e} \cdot \underset{\sim}{\mathbf{F}}^{eT}, \quad J\,\underset{\sim}{\mathbf{s}} = \underset{\sim}{\boldsymbol{\chi}}^{e-T} \cdot \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}^e} \cdot \underset{\sim}{\mathbf{F}}^{eT}$$

$$J\underset{\approx}{\mathbf{M}} = \underset{\sim}{\boldsymbol{\chi}}^{-T} \cdot \rho \frac{\partial \Psi}{\partial \underset{\approx}{\mathbf{K}}^e} : (\underset{\sim}{\boldsymbol{\chi}}^{T} \boxtimes \underset{\sim}{\mathbf{F}}^{T})$$

$$R = \rho \frac{\partial \Psi}{\partial q}, \quad s_0 = -\frac{\partial \psi_0}{\partial T}$$

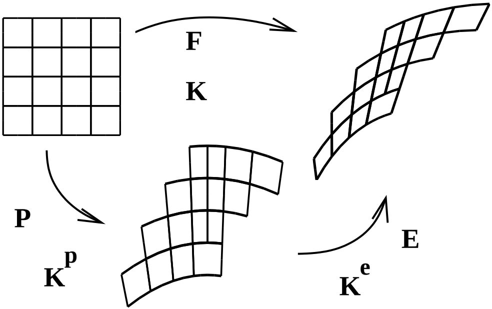
for the additive decomposition of the microdeformation gradient

Finite deformation of elastoviscoplastic micromorphic media

- quasi–additive decomposition of the micro–deformation gradient

$$\underset{\approx}{\mathbf{K}} = \underset{\sim}{\chi}^{p-1}.\underset{\approx}{\mathbf{K}}^e : (\underset{\sim}{\chi}^p \boxtimes \underset{\sim}{\mathbf{F}}^p) + \underset{\approx}{\mathbf{K}}^p$$

according to [Forest and Sievert, 2003]. The objective is here to define a common intermediate configuration simultaneously releasing simple and double stresses, as it will turn out.



Hyperelastic state laws

state laws including hyperelastic relationships

$$J\underset{\sim}{\boldsymbol{\sigma}} = 2\underset{\sim}{\mathbf{F}}^e \cdot \rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{C}}^e} \cdot \underset{\sim}{\mathbf{F}}^{eT}, \quad J\underset{\sim}{\mathbf{s}} = \underset{\sim}{\boldsymbol{\chi}}^{e-T} \cdot \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}^e} \cdot \underset{\sim}{\mathbf{F}}^{eT}$$

$$J\underset{\approx}{\mathbf{M}} = \underset{\sim}{\boldsymbol{\chi}}^{e-T} \cdot \rho \frac{\partial \Psi}{\partial \underset{\approx}{\mathbf{K}}^e} : (\underset{\sim}{\boldsymbol{\chi}}^{eT} \boxtimes \underset{\sim}{\mathbf{F}}^{eT})$$

$$R = \rho \frac{\partial \Psi}{\partial q}, \quad s_0 = -\frac{\partial \psi_0}{\partial T}$$

The quasi–additive decomposition leads to an hyperelastic constitutive equation for the conjugate stress $\underset{\approx}{\mathbf{M}}$ in the current configuration, that has also the same form as for pure hyperelastic behaviour.

Dissipative behaviour

- residual dissipation

$$D = \underset{\sim}{\boldsymbol{\Sigma}} : (\dot{\underset{\sim}{\mathbf{F}}}^p \cdot \underset{\sim}{\mathbf{F}}^{p-1}) + \underset{\sim}{\mathcal{S}} : (\dot{\underset{\sim}{\chi}}^p \cdot \underset{\sim}{\chi}^{p-1}) + \underset{\approx}{\mathbf{M}}_0 \vdots \dot{\underset{\approx}{\mathbf{K}}}^p - R\dot{q} \geq 0$$

generalized Mandel stress tensors

$$\underset{\sim}{\boldsymbol{\Sigma}} = \underset{\sim}{\mathbf{F}}^{eT} \cdot (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}) \cdot \underset{\sim}{\mathbf{F}}^{e-T} = \underset{\sim}{\mathbf{C}}^e \cdot \underset{\sim}{\boldsymbol{\Pi}}^e_{\sigma+s}, \quad J\underset{\sim}{\mathcal{S}} = -\underset{\sim}{\chi}^{eT} \cdot \underset{\sim}{\mathbf{s}} \cdot \underset{\sim}{\chi}^{e-T}$$

$$J\underset{\approx}{\mathcal{M}} = \underset{\sim}{\chi}^T . \underset{\cong}{\mathbf{M}} : (\underset{\sim}{\chi}^{-T} \boxtimes \underset{\sim}{\mathbf{F}}^{-T}) \quad \left(\text{or } J\underset{\approx}{\mathcal{M}} = \underset{\sim}{\chi}^{eT} . \underset{\cong}{\mathbf{M}} : (\underset{\sim}{\chi}^{e-T} \boxtimes \underset{\sim}{\mathbf{F}}^{e-T}) \right)$$

- dissipation potential

$$\Omega(\underset{\sim}{\boldsymbol{\Sigma}}, \quad \underset{\sim}{\mathcal{S}}, \quad \underset{\approx}{\mathbf{S}}_0)$$

$$\dot{\underset{\sim}{\mathbf{F}}}^p . \underset{\sim}{\mathbf{F}}^{p-1} = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\Sigma}}}, \quad \dot{\underset{\sim}{\chi}}^p . \underset{\sim}{\chi}^{p-1} = \frac{\partial \Omega}{\partial \underset{\sim}{\mathcal{S}}}, \quad \dot{\underset{\approx}{\mathbf{K}}}^p = \frac{\partial \Omega}{\partial \underset{\approx}{\mathcal{M}}}, \quad \dot{q} = -\frac{\partial \Omega}{\partial R}$$

Plan

Strain gradient medium at finite deformation

- Strain measures

$$STRAIN = \{ \underset{\sim}{\mathbf{C}} = \underset{\sim}{\mathbf{F}}^T \cdot \underset{\sim}{\mathbf{F}}, \quad \underset{\approx}{\mathbf{K}} = \underset{\sim}{\mathbf{F}}^{-1} \cdot \underset{\sim}{\mathbf{F}} \otimes \boldsymbol{\nabla}_X = \underset{\sim}{\mathbf{F}}^{-1} \cdot \underset{\sim}{\mathbf{F}} \otimes \boldsymbol{\nabla}_X \otimes \boldsymbol{\nabla}_X \}$$

alternative strain gradient measure: $\underset{\approx}{\mathbf{K}} = \underset{\sim}{\mathbf{F}}^T \cdot \underset{\sim}{\mathbf{F}} \otimes \boldsymbol{\nabla}_X$

- Power of internal forces

$$J \underset{\sim}{\boldsymbol{\Sigma}} : (\dot{\underset{\sim}{\mathbf{F}}} \cdot \underset{\sim}{\mathbf{F}}^{-1}) + J \underset{\approx}{\mathbf{M}} \,\vdots\, (\dot{\underset{\sim}{\mathbf{F}}} \cdot \underset{\sim}{\mathbf{F}}^{-1}) \otimes \boldsymbol{\nabla}_x = \underset{\sim}{\boldsymbol{\Pi}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\approx}{\mathbf{M}} \,\vdots\, (\underset{\sim}{\mathbf{F}} \cdot \dot{\underset{\approx}{\mathbf{K}}} : (\underset{\sim}{\mathbf{F}}^{-1} \boxtimes \underset{\sim}{\mathbf{F}}^{-1}))$$

$$= \underset{\sim}{\boldsymbol{\Pi}} : \dot{\underset{\sim}{\mathbf{E}}} + \underset{\approx}{\mathbf{M}}_0 \,\vdots\, \dot{\underset{\approx}{\mathbf{K}}}$$

$$\underset{\sim}{\boldsymbol{\Pi}} = J \underset{\sim}{\mathbf{F}}^{-1} \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot F^{-T}, \quad \underset{\sim}{\mathbf{M}}_0 = J \underset{\sim}{\mathbf{F}}^T \cdot \underset{\approx}{\mathbf{M}} : (\underset{\sim}{\mathbf{F}}^{-T} \boxtimes \underset{\sim}{\mathbf{F}}^{-T})$$

Strain gradient medium at finite deformation

- Partition of deformation

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{F}}^e \cdot \underset{\sim}{\mathbf{F}}^p, \quad \underset{\underset{\sim}{-}}{\mathbf{K}} = \underset{\underset{\sim}{-}}{\mathbf{K}}^e + \underset{\underset{\sim}{-}}{\mathbf{K}}^p$$

or $\underset{\underset{\sim}{-}}{\mathbf{K}} = \underset{\sim}{\mathbf{F}}^{p-1} . \underset{\underset{\sim}{-}}{\mathbf{K}}^e : (\underset{\sim}{\mathbf{F}}^p \boxtimes \underset{\sim}{\mathbf{F}}^p) + \underset{\underset{\sim}{-}}{\mathbf{K}}^p$

- State variables

$$STATE := \{\underset{\sim}{\mathbf{E}}^e, \quad \underset{\underset{\sim}{-}}{\mathbf{K}}^e, \quad q, \quad T\}$$

- Hyperelastic relations

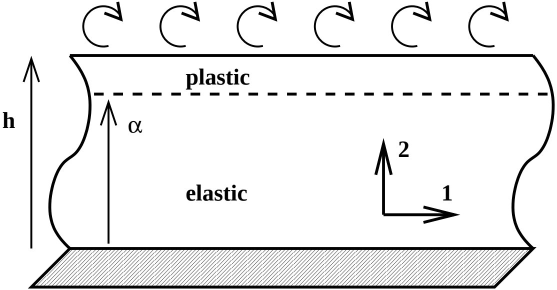
$$J\underset{\sim}{\boldsymbol{\sigma}} = 2\underset{\sim}{\mathbf{F}}^e \cdot \rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{C}}^e} \cdot \underset{\sim}{\mathbf{F}}^{eT}, \quad J\underset{\sim}{\mathbf{s}} = \underset{\sim}{\boldsymbol{\chi}}^{e-T} \cdot \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}^e} \cdot \underset{\sim}{\mathbf{F}}^{eT}$$

$$J\underset{\underset{\sim}{-}}{\mathbf{M}} = \underset{\sim}{\boldsymbol{\chi}}^{-T} \cdot \rho \frac{\partial \Psi}{\partial \underset{\underset{\sim}{-}}{\mathbf{K}}^e} : (\underset{\sim}{\boldsymbol{\chi}}^T \boxtimes \underset{\sim}{\mathbf{F}}^T)$$

$$R = \rho \frac{\partial \Psi}{\partial q}, \quad s_0 = -\frac{\partial \psi_0}{\partial T}$$

Plan

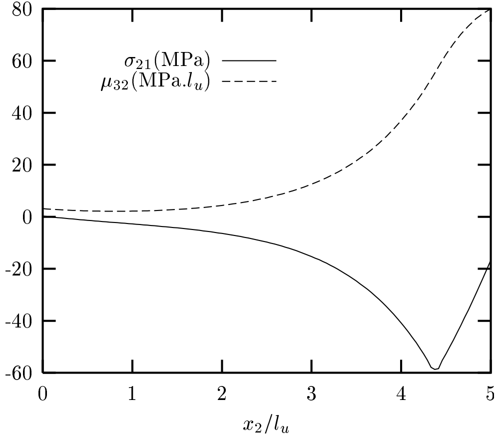
Simple glide problem for the elastic-plastic Cosserat continuum



$$\underline{\mathbf{u}} = \begin{pmatrix} u(x_2) \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{\Phi}} = \begin{pmatrix} 0 \\ 0 \\ \Phi(x_2) \end{pmatrix}$$

$$\underset{\sim}{\mathbf{e}} = \underset{\sim}{\nabla}\underline{\mathbf{u}} + \underset{\approx}{\underline{\epsilon}}.\underline{\mathbf{\Phi}} = \begin{pmatrix} 0 & u' + \Phi & 0 \\ -\Phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \underset{\sim}{\boldsymbol{\kappa}} = \underline{\mathbf{\Phi}} \otimes \underline{\mathbf{\nabla}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Phi' & 0 \end{pmatrix}$$

Simple glide for the elastic-plastic Cosserat medium



A micro–rotation $\Phi = 0.001$ is prescribed at the top $h = 5l_u$. The material parameters are : E $=$ 200000 MPa, $\nu = 0.3$, μ_c=100000 MPa, β=76923 MPa.l_u^2, R_0=100MPa, $a_1 = 1.5, a_2 = 0, b_1 = 1.5 l_u^{-2}, b_2 = 0$. The micro–couple prescribed at the top is $M_{32}^0 = 80$MPa.l_u. l_u is a length unit.

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