Linear elastic trusses leading to continua with exotic mechanical interactions.

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Introduction

Boundary conditions in second gradient or higher order theories

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  These interactions forces being represented by the stress tensor σ (Cauchy theorem).
  When dealing with equilibrium of elastic media, this description can easily be recovered through variational considerations.

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  When dealing with equilibrium of elastic media, this description can easily be recovered through variational considerations.
- Consider for instance a very simple elastic material with elastic energy

$$\tilde{E}(u) = \int_\Omega (A\nabla u)\cdot\nabla u$$

submitted to some volume forces f and surface boundary forces F.
The equilibrium displacement u minimizes $\tilde{E}(u) - \int_\Omega f\cdot u - \int_{\partial\Omega} F\cdot u$.
Setting $\sigma = 2A\nabla u$, the variational formulation reads

$$\forall v,\ \int_\Omega \sigma\cdot\nabla v - \int_\Omega f\cdot v - \int_{\partial\Omega} F\cdot v = 0$$

leading (through an integration by parts) to the PDE formulation

$$div(\sigma) + f = 0 \text{ on } \Omega, \qquad \sigma\cdot n - F = 0 \text{ on } \partial\Omega$$

The last condition being replaced by its dual one $u = 0$ on any part of the boundary wherever the displacement is imposed.

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  Setting $\sigma = 2A\nabla\nabla u$ (a third order tensor) the variational formulation reads

$$\forall v, \int_{\Omega} \sigma \cdot \nabla\nabla v - \int_{\Omega} f \cdot v - \int_{\partial\Omega} F \cdot v = 0$$

  or through two successive integration by parts

$$\forall v, \int_{\Omega} (div(div(\sigma)) - f) \cdot v + \int_{\partial\Omega} (\sigma \cdot n) \cdot \nabla v - (div(\sigma) \cdot n + F) \cdot v = 0$$

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On the boundary, ∇v and v are not independent : the tangent part of the gradient must be eliminated by a new integration by parts. In case of a smooth boundary (edges and wedges are interesting but not considered here) we get

$$\forall v,\ \int_\Omega (div(div(\sigma)) - f) \cdot v + \int_{\partial\Omega} ((\sigma \cdot n) \cdot n) \cdot \frac{\partial v}{\partial n} - (div^s(\sigma \cdot n)_{//} + div(\sigma) \cdot n + F) \cdot v = 0$$

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Leading to the PDE formulation

$$div(div(\sigma)) - f = 0 \text{ on } \Omega, \qquad -div^s(\sigma \cdot n)_{//} - div(\sigma) \cdot n = F \text{ on } \partial\Omega, \qquad (\sigma \cdot n) \cdot n = 0 \text{ on } \partial\Omega$$

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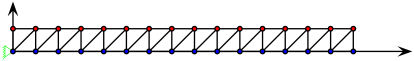
- For higher order materials, more new types of interaction appear.

Discrete systems leading to higher order continua may provide a better understanding of these new mechanical interactions

The flexion beam

Let us begin with a very simple reticulated structure : a beam.



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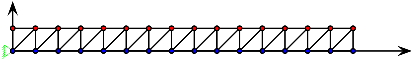

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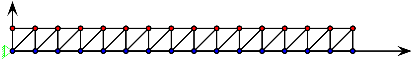

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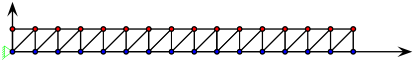

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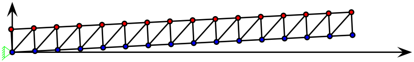

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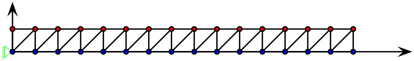

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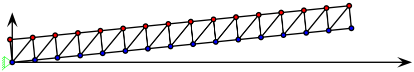

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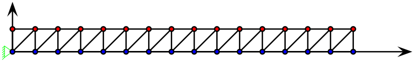

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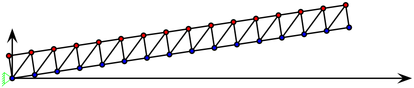

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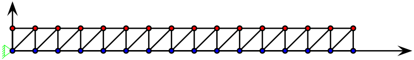

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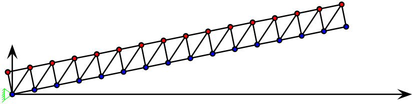

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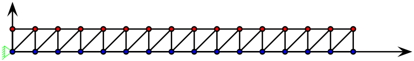

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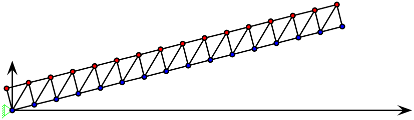
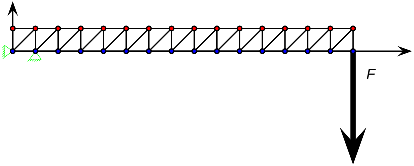


The flexion beam

When fixing an supplementary node, the truss becomes isostatic. At equilibrium it minimizes its potential energy.



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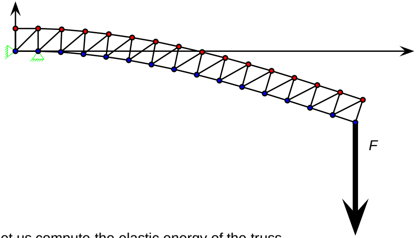
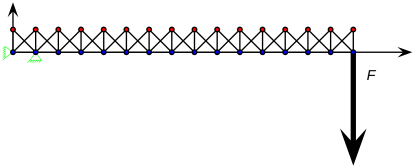


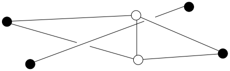
Let us compute the elastic energy of the truss.

The flexion beam

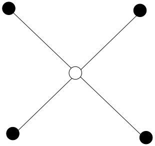
Computation of equilibrium is easier for a slightly different structure:



Remark :
Crossing can be avoided by using a 3D structure

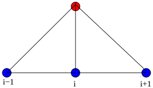


or by adding a internal node at the junction and tuning the stiffnesses of the different bars



We set $x = (x_1, x_2)$.

We denote $u[i] = (u_1[i], u_2[i])$ the displacement of (blue) node i ($i \in \{1, \dots N\}$.
Computation is straightforward if we moreover assume a very high (infinite) stiffness for the horizontal bars. Then $\forall i,\ u_1[i] = 0$.
The elastic energy contained in the structure depends only on the transverse displacement u_2.

In a part 

it reduces to $k(u_2[i-1] - 2u_2[i] + u_2[i+1])^2$
Thus the potential elastic energy of the truss reads

$$E(u) = \sum_{i=2}^{N-1} k(u_2[i-1] - 2u_2[i] + u_2[i+1])^2$$

The flexion beam

- We recognize in $u_2[i-1] - 2u_2[i] + u_2[i+1]$ the finite difference approximation of the second derivative of u_2 with respect to x_1. With a suitable scaling for k, the continuous limit ($N \to \infty$) model reads

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- The equilibrium under the action of a single force F at end point $x_1 = \ell$ minimizes

$$\inf_u \left\{ \tilde{E}(u) - F u_2(\ell) : \; u_1 = 0; \; u_2(0) = 0; \; \frac{\partial u_2}{\partial x_1}(0) = 0 \right\}$$

Hence u_2 satisfies the 4th order differential equation $\left(K u_2''\right)'' = 0$ with the four boundary conditions : fixed displacement $u_2(0) = 0$, applied force $(K u_2'')'(\ell) = F$, fixed rotation $u_2'(0) = 0$ and applied (null) torque $(K u_2'')(\ell) = 0$. Force and torque are dual to displacement and rotation.

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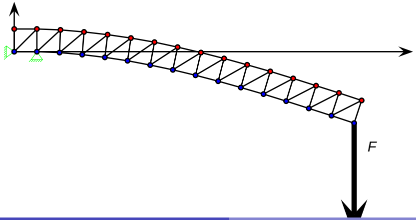
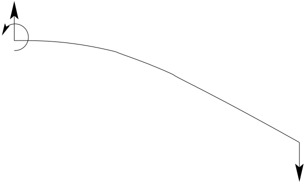
- The solution for the transverse displacement is polynomial : $u_2(x_1) = \frac{-F}{6K\ell}(x_1^3 - 3\ell x_1^2)$.





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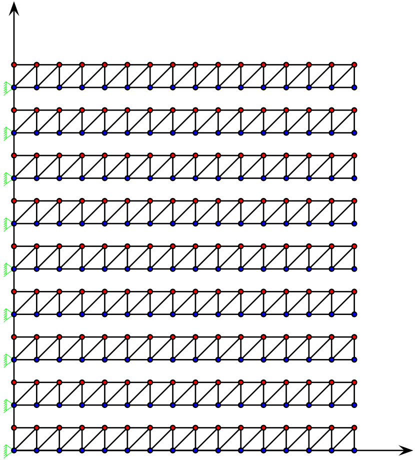

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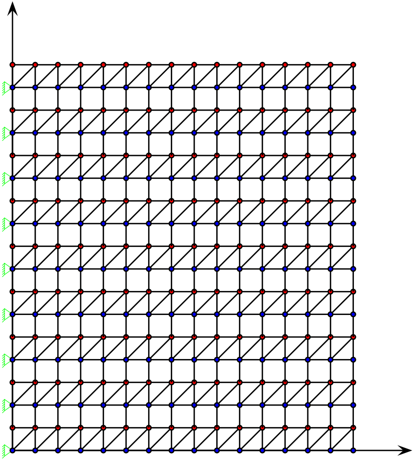

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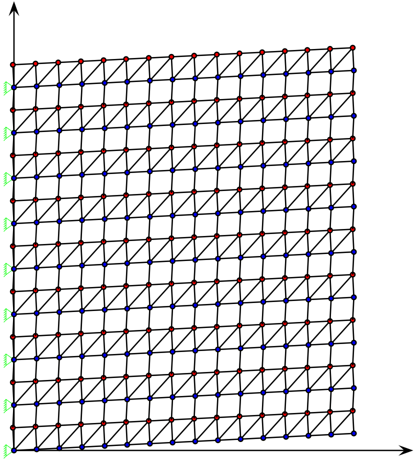

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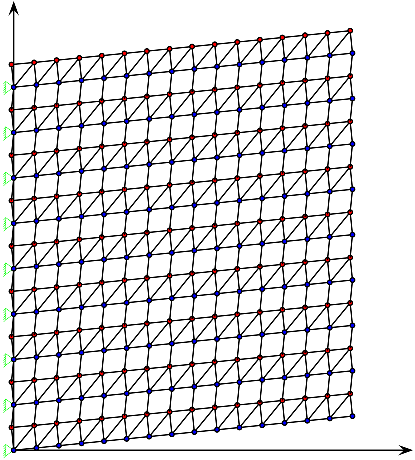

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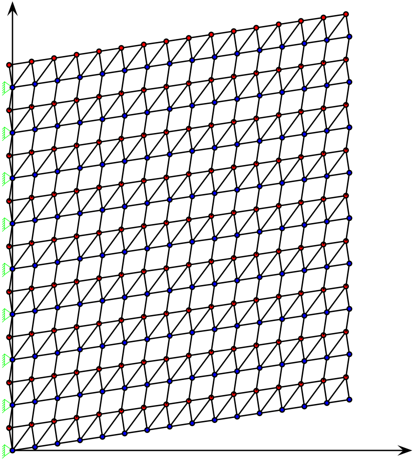

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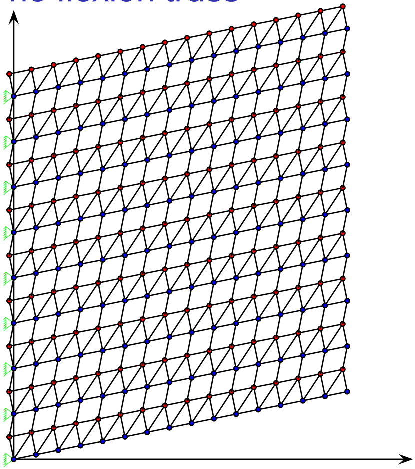

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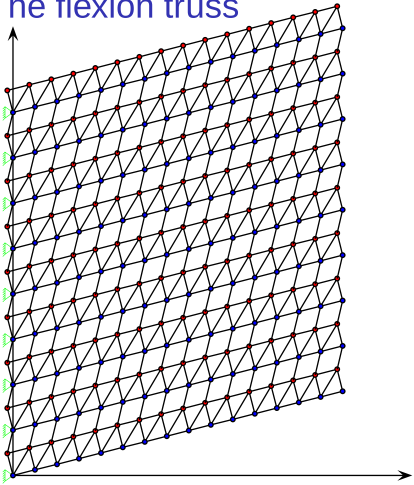

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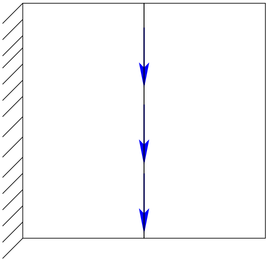
This gives the continuum model (recalling that $u_1 = 0$)

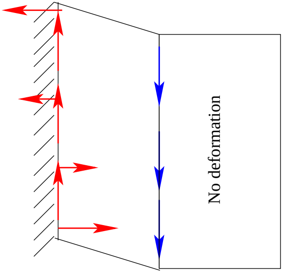
$$\tilde{E}(u) = \int_\Omega K \left(\frac{\partial^2 u_2}{\partial x_1^2} \right)^2 + C \left(\frac{\partial u_2}{\partial x_2} \right)^2$$

The flexion truss

In order to understand the model, let us consider an example of equilibrium :



A surface force is exerted on the middle surface.

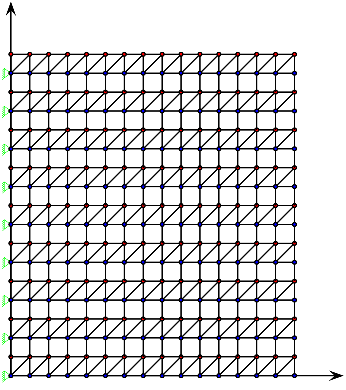


Solution for a classical elastic medium.

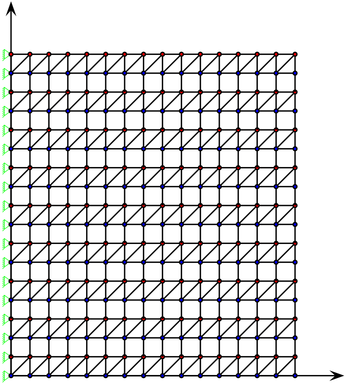
The flexion truss

Effect of the extra boundary condition.

For our truss the solution depends on the extra boundary condition:
- applying a vanishing density of torques $\mathcal{G} = 0$ at $x_1 = 0$ gives no solution : the floppy mode is activated.
- applying a non-vanishing density of torques at $x_1 = 0$, by imposing the dual condition $\partial u_2/\partial x_1 = 0$, gives a solution.

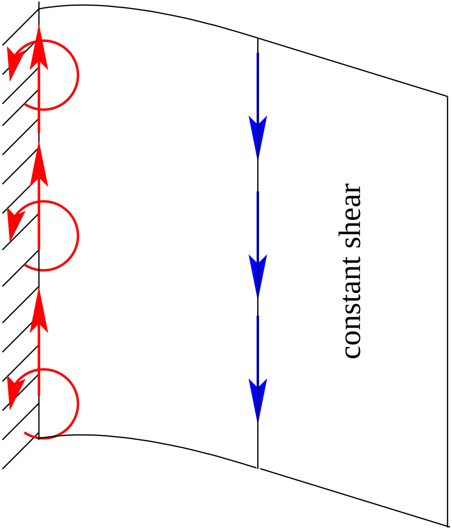


No torque.



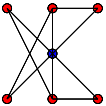
No rotation.

The flexion truss



The pantographic beam

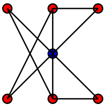
Now we build a structure based on the following basic element,

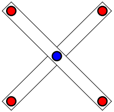


We assume that the bars linking blue and red nodes have very high (infinite) stiffness corresponding to pure flexion (no extension) for the scissors.

The pantographic beam

Now we build a structure based on the following basic element, which has the same behaviour as scissors





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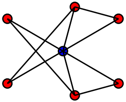

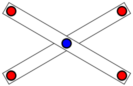

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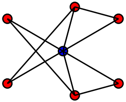

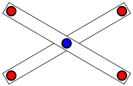

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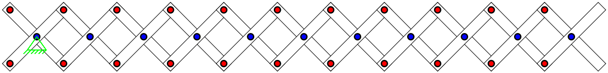
Linking many such cells we get the pantographic structure:



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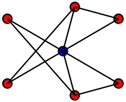

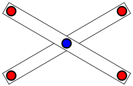

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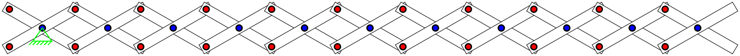


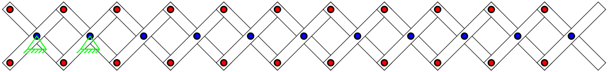
which has an extensional flopping mode (in addition to the rotation).

The pantographic beam

Remarks :

Fixing the deformation of the first cell tends to fix the deformation of all cells.

When fixing the two first (blue) nodes the structure becomes isostatic.



Computing its elastic energy in terms of the displacement of the (blue) nodes is straightforward. We get

$$E(u) = \sum_{i=2}^{N-1} k(u_2[i-1] - 2u_2[i] + u_2[i+1])^2 + k'(u_1[i-1] - 2u_1[i] + u_1[i+1])^2$$

We now recognize in $u_1[i-1] - 2u_1[i] + u_1[i+1]$ the finite difference approximation of the second derivative of u_1 with respect to x_1.
With a suitable scaling for k and k', the continuous limit ($N \to \infty$) model reads

$$\tilde{E}(u) = \int_0^\ell K \left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 + K' \left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2$$

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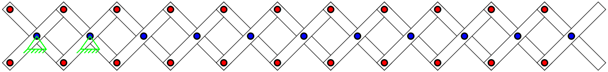

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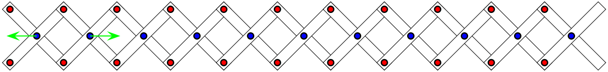
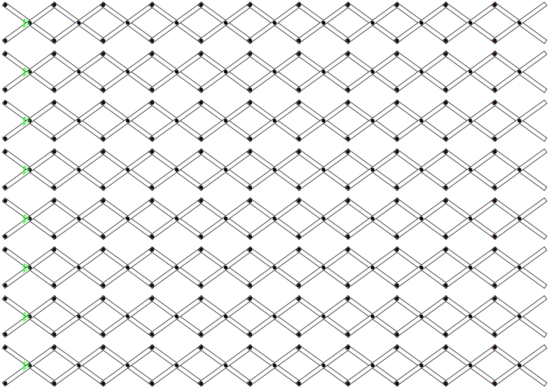
The equilibrium under the action of a single axial force F at end point $x_1 = \ell$ minimizes $\inf_{u_1}\left\{\tilde{E}(u) - Fu_1(\ell) :;\ u_1(0) = 0;\ \frac{\partial u_1}{\partial x_1}(0) = 0\right\}$
Everything can be transposed from the study of the flexion beam to the new beam by replacing the transverse displacement u_2 by the axial one u_1. But the mechanical interpretation is completely different. The action $\mathcal{G} = -(\sigma \cdot n) \cdot n$ corresponds now to a "double force" (or)

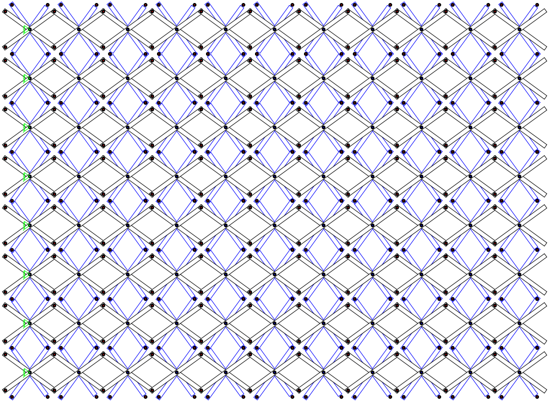


The pantographic truss



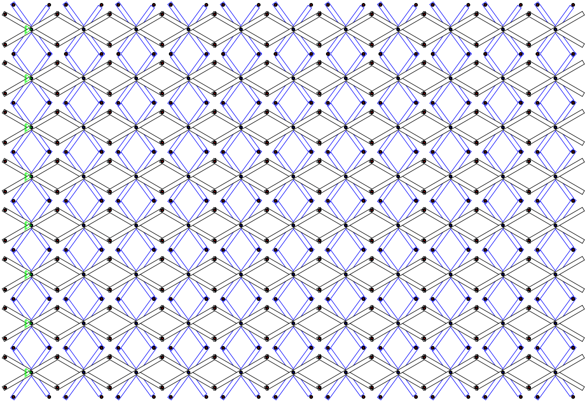
Consider many parallel pantographic beams,

The pantographic truss



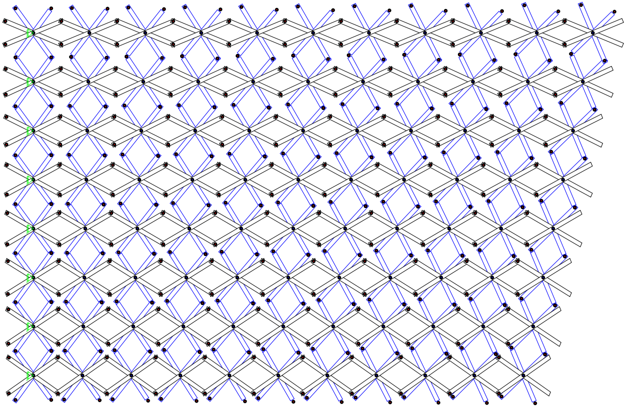
Consider many parallel pantographic beams, link them by pantographic beams,

The pantographic truss



Consider many parallel pantographic beams, link them by pantographic beams, you get a truss with 4 floppy modes.

The pantographic truss



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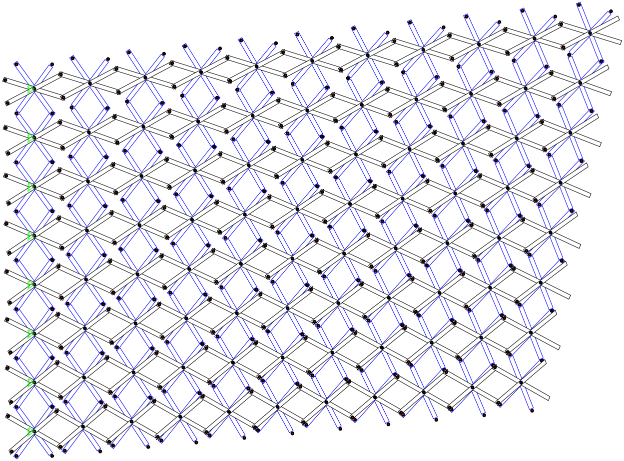

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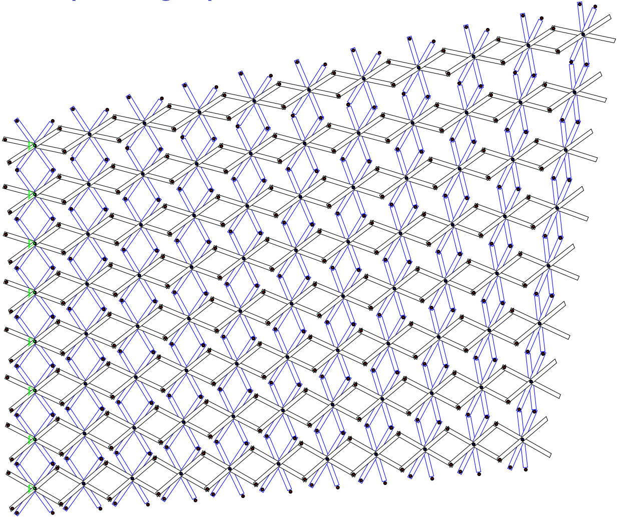

Consider many parallel pantographic beams, link them by pantographic beams, you get a truss with 4 floppy modes.

The pantographic truss

The elastic energy of the structure is the sum of the energies of all pantotographic beams:

$$E(u) = \sum_{j=1}^{M} \sum_{i=2}^{N-1} k(u_1[i-1,j] - 2u_1[i,j] + u_1[i+1,j])^2 + k'(u_2[i-1,j] - 2u_2[i,j] + u_2[i+1,j])^2$$

$$+ \sum_{j=2}^{M-1} \sum_{i=1}^{N} k(u_1[i,j-1] - 2u_1[i,j] + u_1[i,j+1])^2 + k'(u_2[i,j-1] - 2u_2[i,j] + u_2[i,j+1])^2$$

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This gives the continuum model

$$\tilde{E}(u) = \int_\Omega K \left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 + K' \left(\frac{\partial^2 u_2}{\partial x_2^2}\right)^2 + K \left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2 + K' \left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2$$

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The floppy modes are of the form $u(x_1,x_2) = (ax_1x_2 + bx_1 + cx_2 + d, ex_1x_2 + fx_1 + gx_2 + h)$ which, with a Dirichlet condition $u = 0$ on the boundary $x_1 = 0$, reduce to

$$u(x_1,x_2) = ((ax_2 + b)x_1, (ex_2 + f)x_1).$$

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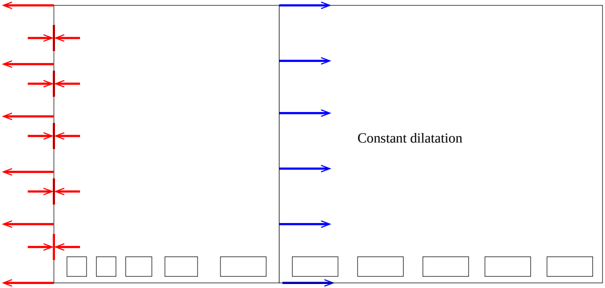
One can still act on the material at the fixed surface $x_1 = 0$ by
- applying a density $\mathcal{G}$ of torques (or fixing the "rotation $\frac{\partial u_2}{\partial x_1}$)
- applying a density $\mathcal{G}$ of "double-forces" (or fixing the "dilatation $\frac{\partial u_1}{\partial x_1}$)

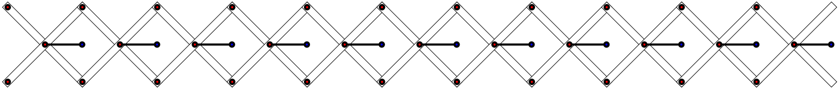
The pantographic truss

An example of equilibrium



The 3rd gradient beam

We first suppress the flexion stiffness of the pantographic structure by considering:



In order to simplify the following drawings, we symbolize it by the triple line 

Its energy is $E(u) = \sum_{i=2}^{N-1} k'(u_1[i-1] - 2u_1[i] + u_1[i+1])^2$ and in the continuous limit $\tilde{E}(u) = \int_0^\ell K' \left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2$.

Then we construct the Warren-type beam (where the upper line is the structure we just described and the other bars are non extensible)

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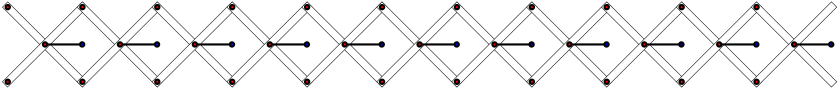

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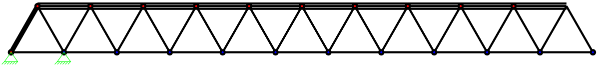

Deformations with constant curvature are the only floppy modes.

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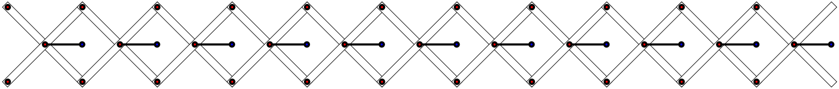

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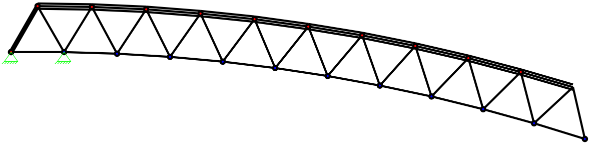

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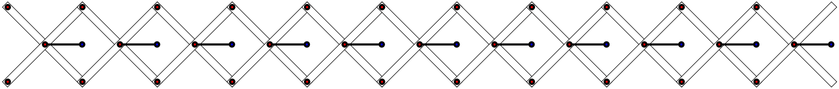

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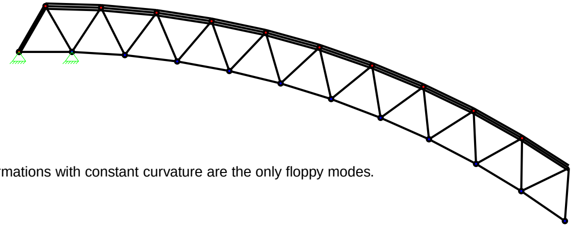

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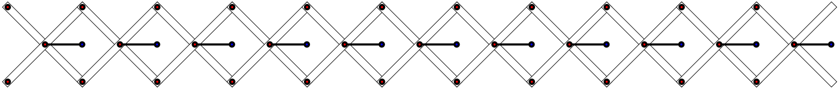

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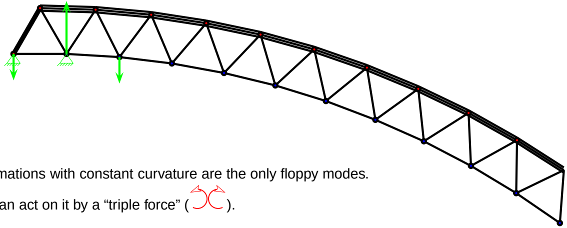


Deformations with constant curvature are the only floppy modes.

One can act on it by a “triple force” ( ).

The 3rd gradient beam

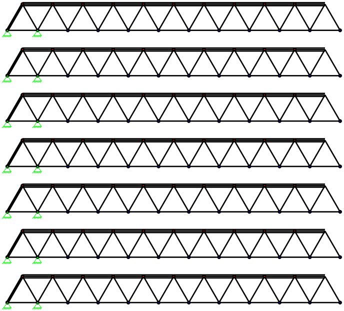
The beam is non extensible : $u_1 = 0$.
Its energy (in terms of the transverse displacement of the blue nodes) reads

$$E(u) = \sum_{i=2}^{N-2} k(-u_2[i-1] + 3u_2[i] - 3u_2[i+1] + u_2[i+2])^2$$

and, in the continuous limit,

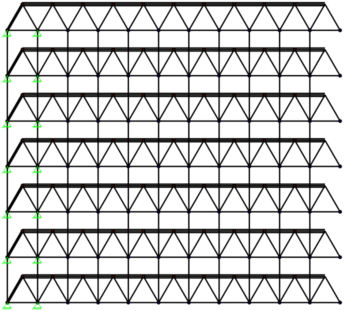
$$\tilde{E}(u) = \int_\Omega K \left(\frac{\partial^3 u_2}{\partial x_1^3} \right)^2$$

The 3rd gradient truss



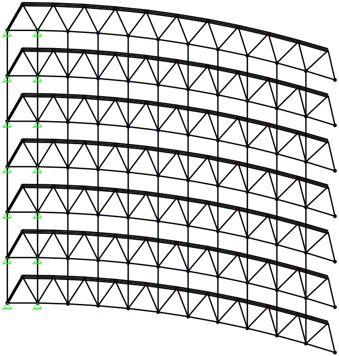
Consider many parallel 3rd gradient beams,

The 3rd gradient truss



Consider many parallel 3rd gradient beams,
link them with bars, you get a truss with one floppy mode.

The 3rd gradient truss



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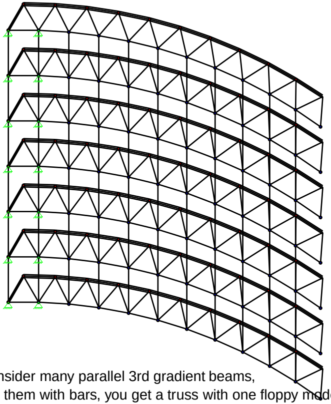
The 3rd gradient truss



Consider many parallel 3rd gradient beams,
link them with bars, you get a truss with one floppy mode.
Its energy reads

$$E(u) = \sum_{j=1}^{M} \sum_{i=2}^{N-2} k(-u_2[i-1,j] + 3u_2[i,j] - 3u_2[i+1,j] + u_2[i+2,j])^2 + \sum_{j=1}^{M-1} \sum_{i=1}^{N} c(u_2[i,j] - u_2[i,j+1])^2$$

In the continuous limit, we get

$$\tilde{E}(u) = \int_\Omega K \left(\frac{\partial^3 u_2}{\partial x_1^3} \right)^2 + C \left(\frac{\partial u_2}{\partial x_2} \right)^2$$

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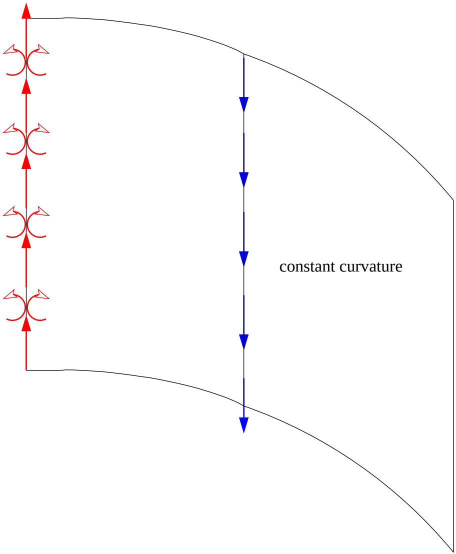
Example of equilibrium :



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- In elasticity, discrete systems can lead to very rich behaviors.
- These behaviors can also be recovered through homogenization procedures. But obtaining them explicitely is a challenge.